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DISTRIBUTION OF THE MEANS DIVIDED BY THE STANDARD DEVIATIONS OF SAMPLES FROM NON-HOMOGENEOUS POPULATIONS

By

G. A. BAKER

In a previous paper[1] the distributions of the means and variances, means squared and variances of samples of two drawn from a non-homogeneous population composed of two normal populations have been discussed. It is the purpose of this paper to discuss similarly the distribution of the means of samples of two measured from the mean of the population divided by the standard deviations of the samples for such parent populations and to present experimental results for samples of four.

CASE $n = 2$

Suppose that a population represented by

(1)

$$f(x) = \frac{N_1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x+m_1)^2}{\sigma_1^2}} + \frac{N_2}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x-m_2)^2}{\sigma_2^2}}$$

$$-N_1 m_1 + N_2 m_2 = 0$$

is considered. If $n-s$ individuals come from the first component and s from the second in drawing samples of n and if $\bar{m}$ is the mean of the sample measured from the mean of the population and $\bar{\sigma}$ is the standard deviation of the sample,[2] then

(2)

$$\frac{\bar{m}}{\bar{\sigma}} = \frac{-(n-s)\,\bar{m}_1 + s\,\bar{m}_2}{\sqrt{n}\left[(n-s)\bar{\sigma}_1^2 + s\bar{\sigma}_2^2 + \frac{(n-s)s}{n}\,(\bar{m}_1 + \bar{m}_2)^2\right]^{\frac{1}{2}}}$$

---

[1] *Annals of Mathematical Statistics*, Vol. 2, No. 3, Aug. 1931.

[2] "Random Sampling from Non-Homogeneous Populations"—*Metron*, Vol. 8, No. 3, p. 6.

where $\bar{m}_1, \bar{m}_2, \bar{\sigma}_1^2$ and $\bar{\sigma}_2^2$ are estimates of the corresponding parameters of (1).

For the case $n = 2$ when both individuals come from the first component of (1) it is known that the distribution of the means divided by the standard deviations of the samples is proportional to

(3)
$$\frac{du}{1+u^2}$$

the origin of u being taken at $-\frac{m_1}{\sigma_1}$. Similarly, when both individuals of the sample come from the second component, $\frac{\bar{m}}{\bar{\sigma}}$ is distributed as proportional to

(4)
$$\frac{dw}{1+w^2}$$

the origin of w being taken at $\frac{m_2}{\sigma_2}$.

When one individual comes from each component (2) becomes

(5)
$$\frac{\bar{m}}{\bar{\sigma}} = \frac{-\bar{m}_1 + \bar{m}_2}{\sqrt{2}\left[\frac{1}{2}(\bar{m}_1 + \bar{m}_2)^2\right]^{\frac{1}{2}}}$$

because no estimate of the standard deviations of the components of (1) can be made from one individual. The distribution of $\bar{m}_1$ is proportional to the first component of (1), and $\bar{m}_2$ is distributed as proportional to the second component. The distributions of $\bar{m}_1$ and $\bar{m}_2$ are independent.

Expression (5) can be rewritten as

(6)
$$\frac{\bar{m}}{\bar{\sigma}} = 1 - \frac{2}{1 + \frac{\bar{m}_2}{\bar{m}_1}}.$$

Put

$$\frac{\bar{m}_2}{\bar{m}_1} = v .$$

(7)

$$\frac{\bar{m}_2}{\bar{m}_1} = v.$$

If the distribution of v is found, the distribution of $z = \frac{\bar{m}}{\sigma}$ may be found by making the transformation

(8)

$$z = 1 - \frac{2}{1+v}$$

or

(9)

$$v = -1 - \frac{2}{z-1}$$

and

(10)

$$dv = \frac{2}{(z-1)^2}\, dz.$$

The distribution of v is a special case of the distribution of an index both of whose components follow the normal law. That is, we seek the distribution of

$$v = \frac{y}{x}$$

x and y being distributed as

(11)

$$z_0 e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(x-\bar{x})^2}{\sigma_1^2} - 2r\frac{(x-\bar{x})}{\sigma_1}\frac{(y-\bar{y})}{\sigma_2} + \frac{(y-\bar{y})^2}{\sigma_2^2}\right]}.$$

This distribution may be obtained as follows.

Lemma I.[2,3] If two variables x and y, $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$

---

[2] (Loc. cit.)

[3] Baten, W. D. "Combining Constant Probability Functions"—*American Mathematical Monthly,* Oct., 1930.

are so related that the probability of an x in dx and of a y in dy is

$$f(x,y)\,dx\,dy$$

then the probability that $v = x - y$ is in dv is proportional to

$$\left[\int_{-\infty}^{\infty} f(v+y,y)\,dy\right]dv.$$

Consider, first, the portion of (11) in the first quadrant. Put

$$v = \frac{y}{x}$$

and take the logarithm of each side, thus

$$\log v = \log y - \log x. \tag{12}$$

Put

$$\begin{aligned} I &= \log v \\ w &= \log y \\ u &= \log x \end{aligned}$$

and (12) becomes

$$I = w - u \tag{13}$$

where the range of w and u is $-\infty$ to $+\infty$. The equation of the correlation surface of w and u is proportional to

(14)

$$F(w,w) = e^u e^w e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(e^u-\bar{x})^2}{\sigma_1^2} - 2r\frac{(e^u-\bar{x})}{\sigma_1}\frac{(e^w-\bar{y})}{\sigma_2} + \frac{(e^w-\bar{y})^2}{\sigma_2^2}\right]}$$

Hence, $F(u, I+u)du$ when the transformation $e^u = x$ is made, becomes

(15)

$$x e^I e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{(x-\bar{x})^2}{\sigma_1^2} - \frac{2r(x-\bar{x})}{\sigma_1}\frac{(xe^I-\bar{y})}{\sigma_2} + \frac{(xe^I-\bar{y})^2}{\sigma_2^2}\right]}dx$$

where x ranges from 0 to ∞. By the application of Lemma I the proportional probability of a value of v in dv when x and y are both positive is obtained by integrating (15) from 0 to ∞ with respect to x and making the transformation $v = e^I$. Thus,

(16)

$$\frac{\sigma_1\sigma_2\sqrt{1-r^2}}{a}\, e^{-\frac{1}{2}\frac{1}{1-r^2}\left[\frac{\bar{x}^2}{\sigma_1^2}-2r\frac{\bar{x}\bar{y}}{\sigma_1\sigma_2}+\frac{\bar{y}^2}{\sigma_2^2}\right]}$$

$$+\frac{b}{a^{3/2}}\, e^{-\frac{(\bar{x}v-\bar{y})^2}{2a}} \int_0^{\frac{b}{\sigma_1\sigma_2\sqrt{1-r^2}\sqrt{a}}} e^{-\frac{1}{2}z^2}\,dz$$

$$+\frac{\sqrt{\pi}}{\sqrt{2}}\,\frac{b}{a^{3/2}}\, e^{-\frac{(\bar{x}v-\bar{y})^2}{2a}}$$

where

$$a=\sigma_2^2-2r\sigma_1\sigma_2 v+\sigma_1^2 v^2$$

$$b=(\sigma_1^2\bar{y}-r\sigma_1\sigma_2\bar{x})v+(\sigma_2^2\bar{x}-r\sigma_1\sigma_2\bar{y})$$

is obtained. The distribution of v if both x and y are negative (and hence v positive) is the same as (16) except that the last term is reversed in sign. Thus, for v positive the distribution of v is proportional to two times the first two terms of (16). If v is negative, i.e. x negative and y positive or x positive and y negative, (16) is obtained in one case and (16) with the sign of the last term changed in the other case. That is, the distribution of v is proportional to the first two terms of (16) when v ranges from $-\infty$ to $+\infty$.

In our case $r=0$, $m_1=\bar{x}$, $m_2=\bar{y}$. Hence, the distribution of v becomes proportional to

(17)

$$\frac{\sigma_1\sigma_2\, e^{-\frac{1}{2}\left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}\right]}}{(\sigma_2^2+\sigma_1^2v^2)}$$

$$+\frac{\sigma_1^2 m_2 v+\sigma_2^2 m_1}{(\sigma_2^2+\sigma_1^2v^2)^{3/2}}\, e^{-\frac{(m_1v-m_2)^2}{2(\sigma_2^2+\sigma_1^2v^2)}} \int_0^{\frac{\sigma_1^2 m_2 v+\sigma_2^2 m_1}{\sigma_1\sigma_2\sqrt{\sigma_2^2+\sigma_1^2v^2}}} e^{-\frac{1}{2}z^2}\,dz.$$

From (8), (9), (10), and (17) z is distributed as proportional to

(18)

$$\frac{\sigma_1\sigma_2}{A} e^{-\frac{1}{2}\left[\frac{m_1^2}{\sigma_1^2}+\frac{m_2^2}{\sigma_2^2}\right]} + \frac{B}{A^{3/2}} e^{-\frac{[z(-m_1+m_2)-(m_1-m_2)]^2}{2A}} \int_0^{\frac{B}{\sigma_1\sigma_2 A^{1/2}}} e^{-\frac{1}{2}u^2}\,du$$

where

$$A = (\sigma_1^2)+\sigma_2^2)z^2+2(\sigma_1^2-\sigma_2^2)z+(\sigma_1^2+\sigma_2^2)$$

and

$$B = z(\sigma_2^2 m_1 - \sigma_1^2 m_2)-(\sigma_1^2 m_2+\sigma_2^2 m_1).$$

The origin for z is at the mean of population (1).

Thus the distribution of $\frac{m}{\sigma}$ for samples of two drawn from a population represented by (1) has been completely determined as being proportional to

(19)

$$k_1(3)+k_2(4)+k_3(18).$$

Let A_1, A_2 and A_3 be the respective areas under the curves represented by the three terms of (19). Then k_1, k_2, and k_3 are to be so determined that

$$A_1+A_2+A_3=N$$

where N is the total number of samples considered, and that

$$\frac{1}{A_1}=\frac{k^2}{A_2}=\frac{2k}{A_3}$$

where

$$k=\frac{N_2}{N_1}.$$

Expression (19) indicates, in general, that if the means of the components do not coincide and if one component is not large compared with the other, both tails of the distribution of the means of samples measured from the mean of the population divided by the standard deviations of the samples are heavier for populations of the type (1) than for normal populations. In case the means of the components coincide one tail will be heavier $(\sigma_1^2 \neq \sigma_2^2)$. In any case at least one tail will be heavier.

EXPERIMENTAL RESULTS

Samples of four were drawn from a population approximately represented by

(1)

$$f(x) = \frac{648}{5\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-15.5)^2}{25}} + \frac{648}{5\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-32.5)^2}{25}}$$

which is the same as Population I in the first reference. These samples were drawn by throwing dice. The means of these samples were calculated and referred to the mean of (1) as an origin. The standard deviations of the samples were obtained and $\frac{\bar{m}}{\sigma}$ calculated. A grouped frequency distribution of 1038 of these values is presented in Chart I and in Table I. Large values are obtained more frequently than would be expected from a normal population.

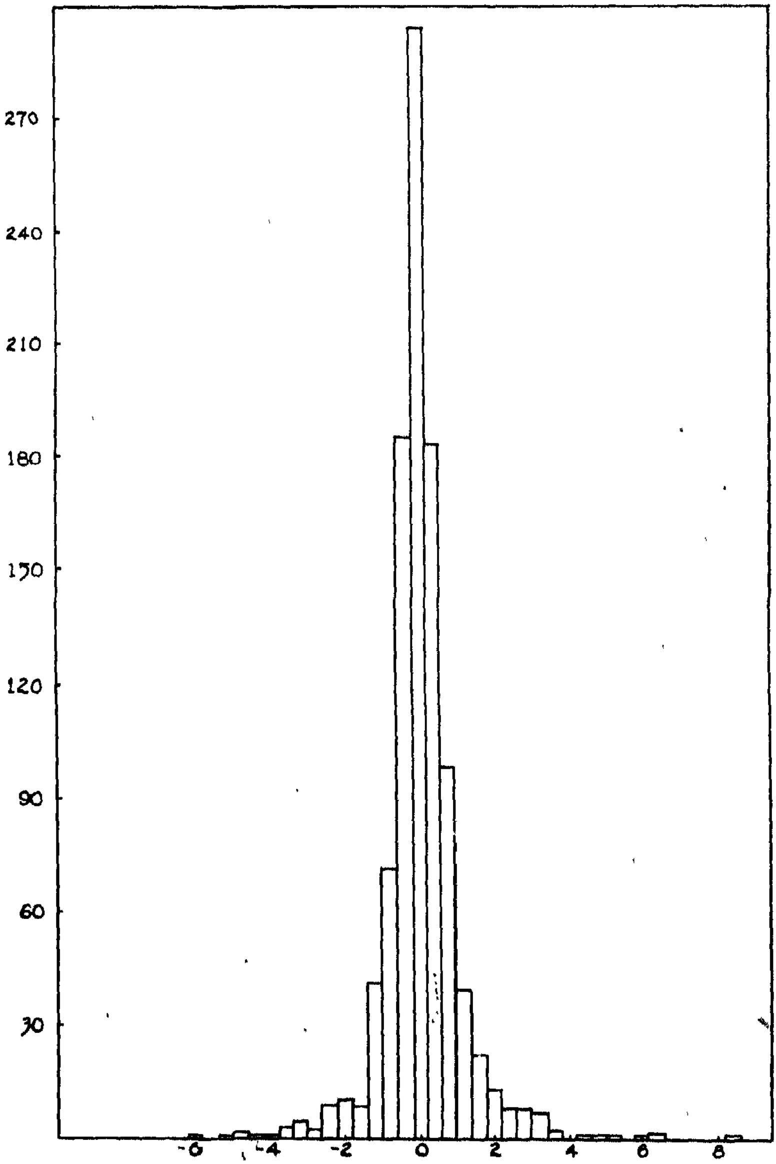
TABLE I.

Grouped Frequency distribution of 1038 Values of $\overline{m}/\sigma$ for Samples of Four from Population I

Middle of Interval	Frequency
—6.0	1
—5.6	0
—5.2	1
—4.8	2
—4.4	1
—4.0	1
—3.6	4
—3.2	6
—2.8	4
—2.4	10
—2.0	11
—1.6	9
—1.2	42
—0.8	71
—0.4	185
0	294
0.4	183
0.8	100
1.2	40
1.6	23
2.0	14
2.4	9
2.8	9
3.2	8
3.6	4
4.0	0
4.4	1
4.8	1
5.2	1
5.6	0
6.0	1
6.4	2
6.8	0
7.2	0
7.6	0
8.0	0
8.4	1

CHART I

Grouped Frequency Distribution of 1038 Values of $\overline{m}/\overline{\sigma}$ for Samples of Four from Population I



A STATISTICAL APPROACH TO MATHEMATICAL FORMULATION OF DEMAND-SUPPLY-PRICE RELATIONSHIPS

By
ROBERT W. BURGESS

A scientific approach to the practical problem of forecasting the prices of commodities clearly requires the development of methods of a somewhat mathematical type for analyzing the relationships between demand, supply, cost, and price. In the case of cotton and other annual crop agricultural commodities, the multiple correlation, link-relative and trend-ratio methods as applied by Moore, Schultz, B. B. Smith, Ezekiel, Holbrook and E. J. Working, and others, have demonstrated their worth. But for copper, lead, rubber, and similar commodities not on an annual crop basis, where quantity produced, or the quantity available, in a given period cannot logically be considered as the supply linked to the average price of that period, the method seems inapplicable and another type of approach is necessary. For this reason, and because of the failure of price to function as expected as a major regulator in our present money economy, it seems worth while to attempt to develop a general mathematical procedure involving cost, demand and supply functions and to analyze elasticity of supply and elasticity of demand as a mathematician naturally does. But some mathematical studies along these lines have not seemed to represent a truly scientific approach to the problem, however helpful they may be in suggesting potentially valuable ideas.

The mathematical economist of the non-statistical type sometimes seems to believe that he has contributed to the solution of economic problems if he finds an answer in the form of a mathematical equation with undetermined constants. The determination of these constants is left as a secondary step to the statistician. But in certain cases, at least, the determination of these constants would be at least as difficult as the original problem. Moreover, the assumptions made in setting up the equations, while not always stated explicitly, have not been shown to constitute a sufficiently close aproximation to actual non-exceptional conditions for the analysis to be useful. It may be suggested that, according to correct scientific procedure, careful statistical analysis of known and knowable conditions is necessary *before* worth while mathematical formulation can be attempted.

To bring out my point of view by discussion of a particular equation, Professor Evans, in his "Introduction to Mathematical Economics," uses as a cost function, that is to say, a function stating total cost for goods produced in a given unit of time, in terms of the quantity produced, u,

$$q(u) = Au^2 + Bu + C$$

This formula involves several unstated assumptions:

(a) That the same continuous formula will apply over the entire range of u which may appropriately be considered in discussing a particular problem. Actually, several points of discontinuity would be more normal. In fact the natural method of statement of this relation might be merely a number of discrete points rather than a function defined for all values of u. For instance, under the conditions of a particular problem, increase in production might be accomplished solely by increasing the number of machines engaged in that process. If so, any rate of output other than an integral multiple of the normal output of the machine might represent inefficient operation and therefore be barred from further discussion. More generally, I should expect a very high unit cost for low values of u, when production is on a job basis, with abrupt downward steps as quantity production methods are applied, and finally a nearly horizontal line.

(b) That a term of the type Au^2 is worth considering in the typical case. Some analysis seems called for as to when this term is appropriate. In mining and agriculture, some such increase of cost with increasing quantity undoubtedly does occur, but it is difficult to imagine practical cases of factory production in which unit cost increases with quantity at any such rate if advance notice of contemplated increase has been given. If production beyond present normal plant capacity is desired, such forced production might involve some temporary increase in unit cost as production increased, but in these days such a case would be unusual.

(c) That the variation of unit cost with quantity is important enough to justify singling that out as the single factor of variation, although actually my impression is that for rather broad ranges of quantity, other factors are more significant causes of variation in costs. Such factors include regularity of production rate, the weather, labor conditions, character of supervision, and management pressure reflecting price conditions for the product.

(d) That $\frac{C}{u}$ is an adequate expression for the element of unit cost which decreases as u increases. Accounting discussions of total cost usually emphasize, in addition to constant overhead, certain elements of cost which increase somewhat with the quantity produced, but not in direct proportion. There is also the very sharp reduction in unit costs when production is first initiated. As a first hypothesis as a basis for detailed statistical study, I suggest that a term of the type Ce^{-ku} might be useful, either alone or with the term $\frac{C}{u}$, to cover the elements of unit costs which decrease as u increases.

(e) That some definition of cost of a logical nature can be framed and applied to this case.

The statistical aproach to a cost function might well show that the range of applicability of a continuous formula is rather narrowly limited and that a careful statement as to attendant conditions is at least as important for reliability of the results as precise determination of the constants. Unfortunately, the results of statistical experiments along these lines are not available. In these days of stiff competition and rigorous government regulation, industrial concerns will be reluctant to permit publication of the kind of analysis of costs which is essential to what seems to be the correct scientific approach to this problem. It is hoped, however, that the preceding discussion has been specific enough to show the nature of the analysis which I think should precede the formulation of an expression of functional relationship, and to suggest the kind of discussion which a mathematical economist should include in his results.

Returning to the general criticism that the treatment of supply-demand-price relationships by certain mathematical economists is not truly scientific, let us review in broad lines the history of scientific progress in those lines where it has admittedly been successful. The steps have been about as follows:

1. Creation of a serviceable mechanism for the measurement of data. In the case of astronomy, for instance, this required the invention of the telescope and general agreement on angular measurement.
2. Careful making and recording of observations with this approved system of measurement, the observations

being in certain cases of events over which the scientist has no control and in other cases of experiments whose conditions could be modified at his convenience.

3. Derivation of empirical laws from these data.
4. Discovery of fundamental principles.

It is true that these steps are not in fact separated as completely as the outline might suggest, and that attempts at the discovery of fundamental principles often help in formulating the plan according to which observations are recorded and give the workers a motive for intensive effort. I think the statement will stand, however, that very few discoveries of fundamental principles have been made until substantial results have been secured under the (1), (2) and (3) headings.

For example, Newton's astronomical laws were discovered after Huyghens and others had made the telescope a useful instrument, Tycho Brahe had made an enormous number of observations, and Kepler had deduced empirical laws for those observations.

Again, in actuarial science, the whole structure of modern life insurance became possible only after careful vital statistics had been recorded for many years and analyzed by the empirical laws of Gompertz and Makeham.

On the other hand, when we turn to treatises on theoretical economics or to books and articles on mathematical economics, there seems to be no trace of the careful recording of observations or their analysis by empirical laws as the basis for their theoretical discussions. I admit, of course, that mathematical formulas are stated which look like empirical laws, but no references are given to any studies justifying these particular formulations. If we imagine ourselves starting the scientific procedure described above as the basis for arriving at real economic laws, we note almost at once that agreement as to the meaning of fundamental terms has not yet been secured. For instance, cost of production is one idea which is fundamental in analysis of demand-supply-price relationships, but cost accountants and economists are by no means agreed among themselves as to what the term should cover. Under the circumstances, it seems to me that the most profitable scientific approach *at present* would be to analyze various relationships which can be put on a quantitative basis, with special attention to noting all the special circumstances of the cases analyzed. For example:

(a) In many cases it would probably be possible to study the relationship between the price of a manufactured article and the price of the raw material or the raw materials used in making it. A simple case which I have actually done in my office and used in price estimating is determining the price of cotton yarn in terms of that of raw cotton. We find it advantageous to compute the cotton yarn price according to the formula, compare that with the actual price and note the relation to general business conditions or to competitive conditions within the industry. The statistical methods required for a problem of this type are obvious, but relatively little, I believe, has been done on this line. Broad comparisons of index numbers of prices of finished goods and raw materials I regard as another kettle of fish altogether.

(b) Relationship of change in price to change in stocks. Any study of actual data of the commodities, such as copper, lead, and rubber, shows that price tends to decline when stocks increase and rise when stocks decrease. A first step in the quantitative approach to price forecasting is to obtain a more precise formulation of this correspondence. It may be noted that some rather vague mathematical ideas come to the surface in discussions of these relationships. For instance, if in a given month production has decreased and consumption has increased, it is sometimes said that these are two arguments for higher prices, and prices are expected to rise. But it may happen in certain cases that even with such a decrease in production and increase in consumption the month's production still exceeds consumption and stocks are increasing. On the whole, then, the monthly figures point to lower rather than higher prices, and it is a useful duty of the mathematician to point this out.

An audience of mathematicians probably regards the preceding illustration as trifling. I bring it up to illustrate the fact that progress toward a more mathematical attitude in commodity forecasting must proceed step by step. A more advanced stage in the quantitative formulation is, of course, to determine the equation connecting change in price and change in stocks as reported monthly. This also has actually proved useful.

(c) Exact definition of the phrase "cost of production" as actually effective when the problem is:

1. Establishing an appropriate price under regulated monopoly conditions.

2. Determining which manufacturing or mining enterprise will survive.

3. Shutting down established sources of supply.

4. Creating new sources of supply.

As I see it, there are at least four costs of production, eacl of which is important under certain circumstances.

1. Complete economic cost, including interest on er tire investment at an appropriate rate.

2. Economic cost, excluding interest on the capital value of ownership.

3. Out-of-pocket expense, which excludes depreciation charges in excess of actual replacements in the period considered, interest on investment, and design or development expenses.

4. Economic cost plus reward for the enterpriser over and above interest at a reasonable rate on his investment.

Roughly speaking, it may be that these are in order closely related to the costs required for the problems stated above.

(d) Analysis of changes in cost at different price levels. One point which has been emphasized by the experience of the past year or two is the fact that costs are by no means kept constant when the price varies. For instance, the best information available two years ago suggested that at 18¢ New York, certain producers of rubber would begin to drop out because of costs above that figure. As a matter of fact, they seem to have been able to change their costs. Such changes are possible by several means, for instance:

1. Wage differentials varying with market price.
2. Bonuses for officials varying with profits.
3. Exploitation of best ores or plantations in times of lower prices.
4. Increased pressure for efficiency when essential.

(e) Careful analysis of the relations of cost to quantity produced with the consideration of:

1. The time ahead that the quantity is known to be required.

2. Continuous production versus intermittent production, perhaps in lots whose size has been determined to

be most economical in view of manufacturing and distributing conditions.

3. Mass production or production in smaller quantities by trained mechanics with attention to the discontinuity in costs per unit when the transition from one type of production to another occurs.

In view of the considerations suggested in (1), (2), and (3), it does not seem to me that the assumption of an algebraic formula connecting cost with quantity produced represents an adequate realistic formulation of the problem.

(f) Analysis of way elasticity of supply or demand actually works. Such an analysis seems to me essential before a definition of coefficient of elasticity is framed or made the basis for elaborate developments. Recent experience has, I think, shown:

1. Elasticity of demand does not mean shrinkage of demand with high prices to anything like the extent expected.

2. Elasticity of supply functions much more slowly than expected at the low-price end of the spectrum. The frictional factors include sympathy with employees who would lose their jobs if economically unprofitable production ceased, reserve funds which permit companies to continue operations when even out-of-pocket expenses are not being fully recovered, bank willingness to lend on commodity stocks which are not, in fact, marketable within a short time, and the cost of shutting down and reassembling a working force.

Most of the matters discussed in (a) to (f) above cannot be analyzed fully on the basis of published records. Moreover, in view of the fact that price information is part of the life blood of any particular business, it will prove difficult for outsiders to secure much of the information which would be needed to complete the analysis. A possible plan would be to place research specialists in industrial or marketing concerns to study actual data. It is not probable that the best research of this type can be done in academic halls as a sideline to teaching. It is also unlikely that corporation officers with an ax to grind can themselves complete the scientific analysis of such material. It seems clear, however, that a satisfactory solution requires a coordination of both points of view.

Summarizing the point of view I have tried to outline, I believe that the analysis of cost-price-supply-demand relationships should be relatively more inductive than it has been, especially in the type of theoretical work classified as mathematical economics. On the other hand, I think that the deductive approach is worth while, that we should try to formulate general principles in this field, and that the ultimate ideal involves a mathematical form,—though, perhaps, when it really covers price situations as they are, a mathematical form somewhat different from those required in the physical, chemical and astronomical sciences.

THE DISTRIBUTIONS OF THE PRECISION CONSTANT AND ITS SQUARE IN SAMPLES OF n FROM A NORMAL POPULATION

By

H. M. Feldman
Washington University

INTRODUCTION

The following paper is a study of the properties of the distributions of the precision constant and its square in samples of from a normal population. The properties studied are (1) modes and optimum values, (2) the first four moments, (3) skewness and flatness, and (4) medians and quartiles.

The distribution curves shown in the figure are for $n=4$, 10, and 25. All the curves are drawn together and to the same scale, so that a graphical comparison of the two distributions can be easily made for both the same and different values of n. The numerical values for the various parameters given in the tables are for $n=4$, 10, 25, and 100, except in the case of the medians and quartiles where the values for $n=100$ are omitted, and in case of $n=4$, no moments higher than the second exist for the precision constant, and none higher than the first for the precision constant squared distribution.

1. *Distributions*

Let us denote the standard deviation, precision constant, and the precision constant squared of the parent population by S, H, and U, respectively, and those of a sample from the given population by s, h, and u, respectively. The standard deviation, s, is then defined in terms of the variates, $x_1, x_2, \cdots, x_n$ and its mean $\bar{x}$ by the equation

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.0)$$

We also have the following well known relations between S, H, and U, or s, h, and u:

$$s = \frac{1}{\sqrt{2h}} \quad \ldots\ldots\ldots\ldots\ldots \quad (1.1)$$

$$s^2 = \frac{1}{2u}, \ or \ u = h^2 \quad \ldots\ldots\ldots \quad (1.2)$$

The distribution of s as given by R. A. Fisher* and others is

$$f(s)ds = \frac{n^{\frac{n-1}{2}} s^{n-2} e^{-\frac{nS^2}{2s^2}} ds}{2^{\frac{n-3}{2}} S^{n-1} \Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots \quad (1.3)$$

where n is the number of items in the sample.

The distribution of s and the transformations (1.1) and (1.2) enable us to find the distributions of the precision constant h, and its square u. Thus, using (1.1) and (1.3) we get

$$F(h)dh = \frac{n^{\frac{n-1}{2}} H^{n-1} h^{-n} e^{-\frac{nH^2}{2h^2}} dh}{2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots \quad (1.4)$$

and by means of (1.2) and (1.3) we find

$$\phi(u)du = \frac{n^{\frac{n-1}{2}} U^{\frac{n-1}{2}} u^{-\frac{(n+1)}{2}} e^{-\frac{nU}{2u}} du}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} \ldots\ldots \quad (1.5)$$

2. *Modal and Optimum Values*

We shall now obtain some of the properties of the distributions of our parameters. The simplest of the properties of any continuous distribution is its modal value or the abscissa of the maximum ordinate of the curve. Denoting the modal values of the distributions (1.4) and (1.5) by $\tilde{h}$ and $\tilde{u}$ respectively, we find from the condition for an extremum

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*See, for example, R. A. Fisher, Applications of "Student's" distribution, Metron, Vol. 5, No. 3, (Dec. 1, 1925), pp. 90-104.

$$\frac{dF}{dh}=0, \quad \frac{d\phi}{du}=0,$$

$$H=\tilde{h} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.0)$$

$$U=\frac{n+1}{n}\tilde{u} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.1)$$

In obtaining $\tilde{h}$ and $\tilde{u}$ we regard h and $\bar{u}$ as variables and H and U as constants. We may, however, reverse our point of view. That is, we regard H and U as variables and h and u as constants. In that case the right hand sides of (1.4) and (1.5) become functions of H and U, and from

$$\frac{dF}{dH}=0, \text{ and } \frac{d\phi}{dU}=0, \text{ we get}$$

$$h=\sqrt{\frac{n}{n-1}}\hat{H} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

$$u=\frac{n}{n-1}\hat{U} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.3)$$

The quantities $\hat{H}$ and $\hat{U}$ R. A. Fisher calls the *optimum* values.

3. *Moments*

Precision Constant

In order to distinguish between the moments of the two distributions treated in this paper, we shall denote the κth moment of the precision constant about the origin by $\mu_i'(n)$ and that of its square by $\mu_i'(u)$ with similar notation for the moments about the mean.

Using the general definition of a moment of a continuous distribution, we obtain for the first moment of n, which is also its mean

$$\mu_1'(h) = \frac{n^{\frac{n-1}{2}} H^{n-1}}{2^{\frac{n-3}{2}}\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty h^{-n} e^{-\frac{nH^2}{2h^2}}\, dh \quad \ldots\ldots (3.10)$$

To put this into an integrable form we make the transformation

$$t = \frac{nH^2}{2h^2} \quad \ldots\ldots\ldots\ldots\ldots (3.11)$$

This yields for the first moment

$$u_1'(h) = H\sqrt{\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \quad \ldots\ldots\ldots\ldots (3.12)$$

To facilitate calculation we express this in terms of factorials. For this purpose we have two cases to consider, namely, the case when n is even, and that when n is odd.

When n is even $\frac{n-2}{2}$ is an integer, and,

$$\begin{aligned}\mu_1'(h) &= H\sqrt{\frac{n}{2}}\,\frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = H\sqrt{n}\,\frac{(n-4)(n-6)\cdots 2}{(n-3)(n-5)\cdots 1}\sqrt{\frac{2}{\pi}}\\ &= H\sqrt{n}\,\frac{\left[2^{\frac{n-4}{2}}\left(\frac{n-4}{2}\right)!\right]^2}{(n-3)!}\sqrt{\frac{2}{\pi}}\end{aligned} \quad \ldots\ldots (3.13)$$

When n is odd $\frac{n-1}{2}$ is an integer and hence

$$\mu_1'(h) = H\sqrt{\frac{n}{2}}\,\frac{\Gamma\,\frac{n-2}{2}}{\Gamma\,\frac{n-1}{2}} = H\sqrt{n}\,\frac{(n-4)(n-6)\cdots 1}{(n-3)(n-5)\cdots 2}\sqrt{\frac{\pi}{2}}$$

$$= H\sqrt{n}\,\frac{(n-3)}{\left[2^{\frac{n-3}{2}}\ \frac{n-3}{2}!\right]}\,2\sqrt{\frac{\pi}{2}} \qquad \ldots\ldots \quad (3.14)$$

Similarly we obtain for the second, third, and forth moments of the distribution of about the origin the following expressions:

$$\mu_2'(h) = H^2\,\frac{n}{2}\,\frac{\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n}{n-3}\,H^2, \quad \ldots\ldots\ldots\ldots \quad (3.15)$$

$$\mu_3'(h) = H^3\left(\frac{n}{2}\right)^{3/2}\frac{\Gamma\left(\frac{n-4}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n}{n-4}\,\mu_1'(h)\,H^3, \quad \ldots\ldots \quad (3.16)$$

$$\mu_4'(h) = H^4\left(\frac{n}{2}\right)^2\,\frac{\Gamma\left(\frac{n-5}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{n^2}{(n-3)(n-5)}\,H^4 \quad \ldots\ldots \quad (3.17)$$

Moments about the Mean

To study such properties of a distribution curve as skewness and flatness we must have the moments of the curve about the mean. To obtain these we use the well known formulae for expressing the moments about the mean in terms of the moments about any origin. Using these formulae we obtain for the first four moments of the precision constant, h, about the mean the following expressions:

$$\mu_1'(h) = 0$$

$$\mu_2(h) = \frac{[n-(n-3)\mu_1'^2(h)]\ H^2}{(n-3)} \quad \ldots\ldots\ldots\ldots\ldots \quad (3.18)$$

$$\mu_3(h) = \frac{[2(n-3)(n-4)\mu_1'^2 - n(2n-9)\mu_1'(n)]\ H^3}{(n-3)(n-4)} \quad \ldots\ldots \quad (3.19)$$

$$\mu_4(h)=\frac{[n^2(n-4)+2n(n-6)(n-5)\mu_1'^2-3(n-3)(n-4)(n-5)\mu'^4]H^4}{(n-3)(n-4)(n-5)}$$

. . . . (3.20)

where $\mu_1'(h)$ is given by (3.12).

For future use we shall give here approximations for $\mu_1'(h)$, $\mu_2(h)$, $\mu_3(h)$, and $\mu_4(h)$. The approximations were obtained by expanding the various quantities into power series of $\frac{1}{n}$. The derivation of these are not difficult but rather long and will therefore not be given in this paper.

These approximations are as follows:

$$\mu_1'(h)=(1+\frac{5}{4n}+\frac{49}{16n^2})H \quad \dots\dots (3.21)$$

$$\mu_2(h)=\frac{1}{2n}H^2 \quad \dots\dots (3.22)$$

$$\mu_3(h)=\frac{a}{n^2}H^3 \quad \dots\dots (3.23)$$

(where a is a constant)

$$\mu_4(h)=\frac{3}{4n^2}H^4 \quad \dots . (3.24)$$

Precision Constant Squared

The first moment of the precision constant squared distribution is, using the general definition of a continuous distribution curve about the origin

$$\mu_1'(u)=\frac{n^{\frac{n-1}{2}}U^{\frac{n-1}{2}}}{2^{\frac{n-3}{2}}\Gamma\frac{n-1}{2}}\int_0^\infty u^{-\frac{n+1}{2}}e^{-\frac{nU}{2u}}du \quad \dots . (3.30)$$

We reduce this to a known integral by means of the transformation

$$t=\frac{nU}{2u} \quad \dots . (3.31)$$

We then obtain for the first four moments of u about the origin the following simple expressions:

$$\mu_1'(u) = \frac{n}{n-3} U \quad \ldots\ldots \quad (3.32)$$

$$\mu_2'(u) = \frac{n^2}{(n-3)(n-5)} U^2 \quad \ldots\ldots \quad (3.33)$$

$$\mu_3'(u) = \frac{n^3}{(n-3)(n-5)(n-7)} U^3 \quad \ldots\ldots \quad (3.34)$$

$$\mu_4'(u) = \frac{n^4}{(n-3)(n-5)(n-7)(n-9)} U^4 \quad \ldots\ldots \quad (3.35)$$

Moments about the Mean

For the moments about the mean of the precision constant squared distribution we have:

$$\mu_1 = 0$$

$$\mu_2 = \frac{2n^2}{(n-3)^2(n-5)} U \quad \ldots\ldots \quad (3.36)$$

$$\mu_3 = \frac{16n^3}{(n-3)^3(n-5)(n-7)} U^3 \quad \ldots\ldots \quad (3.37)$$

$$\mu_4 = \frac{4n^4(2n+27)}{(n-3)^4(n-5)(n-7)(n-9)} U^4 \quad \ldots\ldots \quad (3.38)$$

Skewness and Flatness

From the above expressions for the mean and also from the numerical values given in the tables we may conclude that the precision constant distribution is less skew than the distribution of the precision constant squared, at least for values of n up to 100. But what happens when n grows very large? To answer this we make use of the Pearsonian measure of skewness, β_1, defined by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \ldots\ldots \quad (3.40)$$

Since we are now interested in large values of n we make use of the approximate values of $\mu_2(h)$, and $\mu_3(h)$, which are

$$\mu_2(h) = \frac{1}{2n} H^2, \text{ and } \mu_3(h) = \frac{a}{n^2}$$

Hence we get

$$\lim_{n \doteq \infty} \beta_1(h) = \lim_{n \doteq \infty} \frac{a^2H^6}{n^4} \div \frac{1}{n^3} H^6 = \lim_{n \doteq \infty} \frac{a^2}{n}$$

and $$\lim_{n \doteq \infty} \frac{a^2}{n} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.41)$$

To find $\lim_{n \doteq \infty} \beta_1(u)$ we make use of the exact values of $\mu_2(u)$ and $\mu_3(u)$ which are given by (3.36) and (3.37). This gives

$$\lim_{n \doteq \infty} \beta_1(u) = \lim_{n \doteq \infty} \left[\frac{16n^3U^3}{(n-3)^3(n-5)(n-7)}\right]^2 \div \left[\frac{2n^2U^2}{(n-3)^2(n-5)}\right]^3$$

$$= \lim_{n \doteq \infty} \frac{32(n-5)}{(n-7)^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.42)$$

From (3.41) and (3.42) we learn that both the precision constant distribution, and that of its square, approach perfect symmetry as the size of the sample, n, approaches infinity.

The flatness or kurtosis of a curve is measured by the quantity

$$\beta_2 = \frac{\mu_4}{\mu_1^2} \quad \ldots\ldots\ldots\ldots \quad (3.43)$$

From this and the expressions (3.22 and (3.24), (3.33) and (3.35) we get

$$\lim_{n \doteq \infty} \beta_2(h) = \lim_{n \doteq \infty} \frac{3H^4}{4n^2} \div \left(\frac{H^2}{2n}\right)^2 = 3 \quad \ldots\ldots\ldots\ldots \quad (3.44)$$

and

$$\lim_{n \doteq \infty} \beta_2(u) = \lim_{n \doteq \infty} \frac{4n^4(2n+27)U^4}{(n-3)^4(n-5)(n-7)(n-9)} \div \frac{4n^4U}{(n-3)^4(n-5)}$$

$$= \lim_{n \doteq \infty} \frac{(n-5)(2n+27)}{(n-7)(n-9)} = 2 \quad \ldots\ldots\ldots \quad (3.45)$$

We may conclude, then, that while the distributions of the precision constant and its square are both perfectly symmetrical for very large values of n, they are nevertheless entirely distinct distribution curves for both small and large values of n, since $\lim_{n \doteq \infty} \beta_2(h) = 3$, and $\lim_{n \doteq \infty} \beta_2(u) = 2$. As a matter of fact the distribution of the precision constant approaches the normal curve, while the precision constant squared distribution approaches a curve of the form

$$y = y_0 \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}$$

where y_0 and a are constants.

4. *Quartiles and Medians*

The quartiles of a continuous distribution $f(x)$ may be defined by the equation

$$\int_0^{Q_i} f(x)\,dx = \frac{i}{4}, \quad (i = 1, 2, 3) \quad \ldots\ldots \quad (4.00)$$

For $i = 1$, Q_i is called the lower quartile; for $i = 2$, Q_i is called the median and for $i = 3$, Q_i is called the upper quartile.

In order to find the quartiles of the distributions studied in this paper we must make use of the incomplete Γ- function. This function is defined as follows

$$I(u,p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{1+p}} e^{-v} v^p \, dv \quad \ldots\ldots \quad (4.10)$$

Pearson's "Tables of the Inconplete Γ-Function" give the values of I, u, and p. Thus, if we know any two of the variables we can easily find the third.

Let us take, now, the distribution of h,

$$F(h)dh = \frac{n^{\frac{n-1}{2}} H^{n-1} h^{-n} e^{-\frac{nH^2}{2h^2}} dh}{2^{\frac{n-3}{2}} \Gamma\left(\frac{n-1}{2}\right)}$$

The various quartiles of this distribution will be given by

$$n^{\frac{n-1}{2}} H^{n-1} \int_0^{Q_i} h^{-n} e^{-\frac{nH^2}{2h^2}} dh = \frac{i}{4} \quad \ldots\ldots \quad (4.11)$$

By making the transformation

$$v = \frac{nH^2}{2n^2}$$

(4.11) is reduced to

$$\frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_\infty^{u_i \sqrt{\frac{n-1}{2}}} e^{-v} v^{\frac{n-3}{2}} = \frac{i}{4} \quad \ldots\ldots\ldots \quad (4.13)$$

This is, of course, $I(u_i, \frac{u-3}{2})$, and since i and u are known we can easily find u_i from Pearson's Tables.

Comparing (4.11) and (4.13) and taking into account the transformation (4.12) we find that

$$Q_i = \frac{n^{\frac{1}{2}}}{u_i^{\frac{1}{2}} (2n-2)^{\frac{1}{4}}} \cdot H.$$

Since the lower limit in (4.13) is ∞ instead of 0 we see that the lower and upper quartiles are reversed.

To find the quartiles for the distribution of u we simply make use of the relation $u = h^2$.

In conclusion we may state that the results of this paper are similar to those of Professor Rietz on the distributions of the standard deviation and the variance.*

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*Rietz, H. L. A comparison of the distributions curves of variance and of standard deviation, *Mathematical Monthly*, Vol. 36 (August-Sept., 1929), p. 355.

TABLE I

Modal and optimum values for the distribution of the precision constant and its square from a normal population for samples of 4, 10, 25, and 100.

	Pr. const. h		Pr. const. squared u	
n	Mode	Optimum	Mode	Optimum
4	H	$1.154H$	$0.800\,U$	$1.333\,U$
10	H	$1.054\,H$	$0.909\,U$	$1.111\,U$
25	H	$1.021\,H$	$0.962\,U$	$1.042\,U$
100	H	$1.005\,H$	$0.990\,U$	$1.010\,U$

TABLE II

Values of the mean and the first four moments about the mean of the distributions of the precision constant and its square for samples of 4, 10, 25, and 100 from a normal population.

Precision constant h				
n	Mean μ_1'	μ_2	μ_3	μ_4
4	$1.596H$	$1.454\ H^2$		
10	$1.153\,H$	$0.0982\,H^2$	$0.0678\ H^3$	$0.0815\ H^4$
25	$1.054\,H$	$0.0255\,H^2$	$0.0031\ H^3$	$0.0026\ H^4$
100	$1.013\,H$	$0.0053\,H^2$	$0.000141H^3$	$0.0000907\,H^4$

Precision constant square U				
n	μ_1'	μ_2	μ_3	μ_4
4	$4.000\,U$			
10	$1.429\,U$	$0.816\,U^2$	$3.110\ U^3$	$52.200\ U^4$
25	$1.136\,U$	$0.129\,U^2$	$0.06522U^3$	$0.892\ U^4$
100	$1.031\,U$	$0.224\,U^2$	$0.0199\ U^3$	$0.00129U^4$

TABLE III

Values of β_1 and β_2 for the distribution of the precision constant and its square for samples of 10, 25, 100, and ∞ from a normal population.

	h		u	
n	β_1	β_2	β_1	β_2
10	4.8548	8.4521	17.7807	78.3416
25	0.7596	3.9982	2.0099	5.3172
100	0.00134	3.2289	0.3515	2.5476
∞	0.0000	3.0000	0.0000	2.0000

TABLE IV

Medians and quartiles for the distributions of the precision constant and its square for samples of 4, 10, and 25 from a normal population.

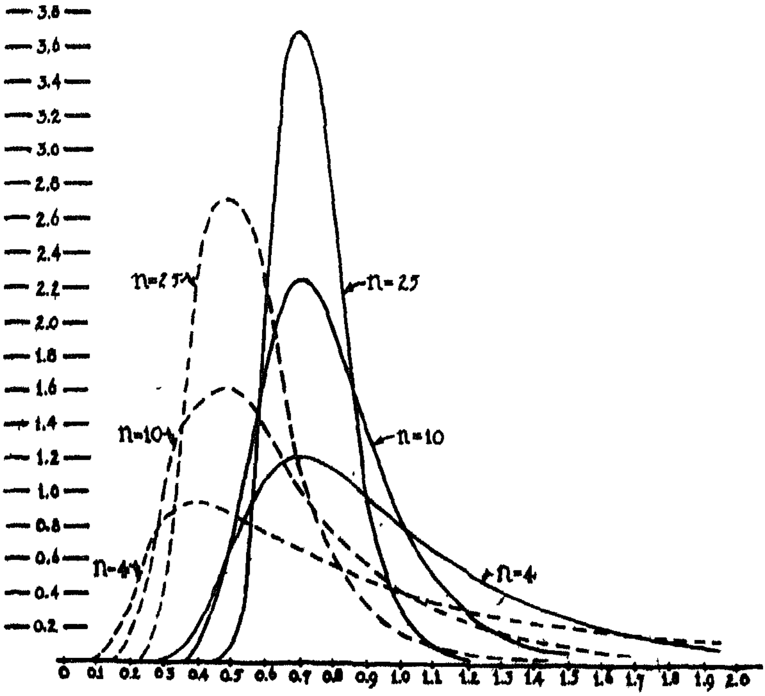
	Precision constant h			Precision constant squared u		
n	Q_1	Median	Q_3	Q_1	Median	Q_3
4	0.985 H	1.298 H	1.816 H	0.970 U	1.685 U	3.298 U
10	0.937 H	1.094 H	1.230 H	0.878 U	1.197 U	1.513 U
25	0.941 H	1.036 H	1.146 H	0.885 U	1.073 U	1.313 U

Distribution curves of the precision constant and its square for samples of 4, 10, and 25, from a normal population.

Note that the mode for the precision constant distribution is the same for all values of n as is seen from (2.0).

The solid curves are for the precision constant, the dotted curves for its square.

The unit used is the standard deviation of the population



A POSTULATE FOR OBSERVATIONS

By R. HENDERSON

When measurements are made by a given observer using a particular instrument, if the mean or expected result of the measurements is not supposed to be equal to the true value of the quantity measured the difference is considered to be an error, personal or instrumental or both. A correction is therefore applied to any such measurement in order to remove the discrepancy. In other words a given combination of observer and instrument is not considered to give correct or balanced measurements until this discrepancy is removed. Also as between two instruments or observers, both giving balanced measurements after the application of known corrections, preference is given to the one which shows the smaller variations between different measurements of the same quantity.

In selecting the formula to be used to determine, from the results of a series of measurements involving certain unknown quantities, the best measures of those quantities we are in a position similar to that of an observer desiring to make a certain measurement and selecting the best available instrument for the purpose. Such an observer would, in the first place, require that the instrument should give balanced measurements and would, in the second place, among a number of such instruments select the one showing the smallest standard deviation. This suggests the following definition and postulate.

Definition A balanced measure of a quantity is one of which the mean or expected value is equal to the true value of the quantity measured.

Postulate Of two or more balanced measures of a quantity the best measure is the one which has the smallest standard deviation.

Repeated Measurements

When we have a number of different results

$$a_1,\ a_2,\ a_3\ \cdots\cdots a_n$$

of balanced measurements of the same quantity a, then any function of the form

$$\frac{\ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_n a_n}{\ell_1 + \ell_2 + \cdots \ell_n}$$

will also be a balanced measure of the quantity. If the standard deviations of the individual measures are respectively

$$\sigma_1,\ \sigma_2\ \cdots\cdot \sigma_n,$$

the square of the standard deviation of the function will be

$$\frac{\ell_1^2\sigma_1^2 + \ell_2^2\sigma_2^2 + \cdots\cdot \ell_n^2\sigma_n^2}{(\ell_1 + \ell_2 + \cdots \ell_n)^2}$$

This is a minimum when $\ell_1\sigma_1^2 = \ell_2\sigma_2^2 = \cdots\cdot = \ell_n\sigma_n^2$ so that the best measure is the average of the individual measures each weighted inversely as the square of its standard deviation. If the individual measures have the same standard deviation this reduces to the ordinary arithmetical average.

Combinations of Observations.

In applying this postulate to the theory of combination of observations, suppose that there are n unknown quantities

$x_i (i = 1,2,3 \cdots\cdot n$ and that we have balanced measures of

$m\ (m > n)$ linear functions of these unknowns of the form

$$y_h = \sum_{i=1}^{i=n} a_{hi}\, x_i \ (h = 1,2,3 \cdots m) \qquad (1)$$

For simplicity we shall assume that the functions have been so taken that the standard deviations of these measures are all equal. Then a linear function of these measures of the form.

$$\sum_{h=1}^{h=m} b_{hj} y_h = \sum_{h=1}^{h=m} \sum_{i=1}^{i=n} b_{hj} a_{hi} x_i \qquad (2)$$

will be a balanced measure of x_j if

$$\sum_{h=1}^{h=m} b_{hj} a_{hi} = \delta_j^2 \qquad (3)$$

where $\delta_j^2 = 1$ if $i = j$ and $\delta_j^2 = 0$ if $i \neq j$

It will be the best measure of that type if $\sum_{h=1}^{h=m} b_{hj}^2$ is a minimum subject to those conditions.

By the method of indeterminate coefficients we find that this occurs when we can write

$$b_{hj} = \sum_{k=1}^{k=n} t_{jk} a_{hk} \qquad (4)$$

and the values of the n^2 coefficients t_{jk} are determined from the n^2 conditions (3). Then

$$x_j = \sum_{k=1}^{k=n} \sum_{h=1}^{h=m} t_{jk} a_{hk} y_h = \sum_{k=1}^{k=n} t_{jk} y_k \qquad (5)$$

if we write y_k for $\sum_{h=1}^{h=m} a_{hk} y_h$. The value of each of the n unknowns x_j is thus expressed in terms of the n functions y_k or, in other words, may be determined from the n equations expressing y_k in terms of the n unknowns x_j These equations take the form, if we write

$$A_{jk} \text{ for} \sum_{h=1}^{h=m} a_{hj}\, a_{hk}$$

$$\sum_{j=1}^{j=n} A_{jk}\, x_j = y_k \qquad (6)$$

It will be noted that although no assumption regarding the law of error, other than that of balance, has been made the equations deduced are the same as those derived, in the ordinary theory of least squares, from the assumption of the normal exponential law.

Measurement of Probabilities.

Where the quantity measured is a probability and the measure is to be determined from the observed result of a finite number of trials we know that, if the probability is p the number of trials n and the number of occurrences of the particular result r then the expected value of r is np. Consequently r/n is a balanced measure of p . The measure usually associated with Bayes' theorem, namely $\frac{r+1}{n+2}$ is not a balanced measure. Its mean value is $p + \frac{1-2p}{n+2}$ which is not equal to p unless p happens to be equal to ½.

For this case a different postulate might consistently with the general methods of science, have been proposed as follows. *That hypothesis is to be adopted which makes the compound probability of the hypothesis and the observed facts a maximum.* If then we considered one value of the probability as likely as another this would mean selecting the value of p which would make $p^r(1-p)^{n-r}$ a maximum. This would have given r/n as before.

Frequency Distributions

The notation on the subject of moments is so unsettled that it appears to be necessary for each writer to specify the notation adopted. In this paper the rth moment about the origin in a finite sample will be written m_r and the corresponding moment about the mean value will be designated by μ_r. The moments in the population from which the sample is drawn will be written $\bar{m}_r$ and $\bar{\mu}_r$ respectively.

In this connection an important consideration arises from the fact that balanced measures are not always consistent under ordinary mathematical transformations. This happens because if y is a balanced measure of x then $f(y)$ is not necessarily a balanced measure of $f(x)$ Let $y = x + h$ and let mean values be indicated by prefixing $\bar{m}_1$ so that

$$\bar{m}_1(h) = \bar{m}_1(y) - x = 0.$$

Then since

$$f(y) = f(h) + hf'(x) + \frac{h^2}{2} f''(x) + etc$$

we have

$$\bar{m}_1 \{f(y)\} = f(x) + \frac{\bar{m}_1(h^2)}{2} f''(x) + c = f(x) + \frac{\bar{\mu}_2}{2} f''(x) + etc$$

Ordinarily therefore unless $f(x)$ is a linear function of x or, if not, y is an exact measure of x, $\bar{m}_1 \{ f(y) \}$ will not be equal to $f(x)$.

A simple illustration of this fact arises in connection with the determination, from a sample, of a measure for $\bar{\mu}_2$. By ordinary transformations we have the well known formula $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$ Also m_2 and m_1 are balanced measure of $\bar{m}_2$ and $\bar{m}_1$ respectively but m_1^2 is not a balanced measure of $\bar{m}_1^2$. We have in fact. $\bar{m}_1(m_1^2) = \bar{m}_1^2 + \frac{1}{n}\bar{\mu}_2$. Therefore,

$$\bar{m}_1(\mu_2) = \bar{m}_1(m_2 - m_1^2) = \bar{m}_2 - \bar{m}_1^2 - \frac{1}{n}\bar{\mu}_2 = (1 - \frac{1}{n})\bar{\mu}_2.$$

The balanced measure of $\bar{\mu}_2$ would therefore be $\frac{n}{n-1}\mu_2$ which is not formally consistent with the balanced measures of

$\bar{m}_1$ and $\bar{m}_2$ in the light of the equation $\bar{\mu}_2 = \bar{m}_2 - \bar{m}_1^2$. By a similar line of reasoning as shown by Thiele, we obtain $\frac{n^2}{(n-1)(n-2)}\mu_3$ as a balanced measure of $\bar{\mu}_3$ and by Tschuprow's modification of Thiele's analysis

$$\frac{n}{(n-1)(n-2)(n-3)}\left\{(n^2-2n+3)\mu_4 - 3(2u-3)\mu_2^2\right\}$$

as a balanced measure of $\bar{\mu}_4$. Here however we are faced with the further difficulty that while this is a balanced measure, its standard deviation for small values of n is so great that possible values of μ_4 and μ_2 would result in negative values of $\bar{\mu}_4$ whereas in any real frequency distribution not only must $\bar{\mu}_4$ be positive but $(\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$ which is the mean value of $\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ must also be positive.

If, therefore, we wish to derive a value of $\bar{\mu}_4$ certainly satisfying this condition we must determine the average value of $\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}$ for all combinations of three values from the sample of n and use it as a balanced measure of $(\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3)$ This average value is found to be $\frac{n^2}{(n-1)(n-2)}(\mu_2\mu_4 - \mu_3^2 - \mu_2^3)$.

The analysis is as follows.

$$\bar{m}_1\left\{\frac{(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2}{6}\right\}$$

$$= \bar{m}_1\{x_1^4x_2^2 + 2x_1^3x_2^2x_3 - x_1^4x_2x_3 - x_1^3x_2^3 - x_1^2x_2^2x_3^2\}$$

$$= \bar{m}_4\bar{m}_2 + 2\bar{m}_3\bar{m}_2\bar{m}_1 - \bar{m}_4\bar{m}_1^2 - \bar{m}_3^2 - \bar{m}_2^3$$

$$= \bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3$$

If repetitions were allowed in the finite sample the average value would be the corresponding expression in moments of the sample, namely, $\mu_2\mu_4 - \mu_3^2 - \mu_2^3$ But since the expression vanishes if two or more of the values of x involved are equal the exclusion of repetitions reduces the total number of permutations three at a time from n^3 to $n(n-1)(n-2)$ without reducing the sum of the values. The average is thereincreased to

$$\frac{n^2}{(n-1)(n-2)}\left(\mu_2\mu_4 - \mu_3^2 - \mu_2^3\right)$$

The second moment $\bar{\mu}_2$ might have been similarly derived as the mean value of $\frac{(x_1 - x_2)^2}{2}$ and the third moment $\bar{\mu}_3$ as the mean value of

$$\frac{(2x_1 - x_2 - x_3)(2x_2 - x_3 - x_1)(2x_3 - x_1 - x_2)}{6}$$

In this latter case the expressions averaged do not vanish when two only of the value of x are equal but they cancel one another in pairs so that their sum vanishes.

We have thus as working approximations

$$\bar{\mu}_2 = \frac{n}{n-1}\mu_2$$

$$\bar{\mu}_3 = \frac{n^2}{(n-1)(n-2)}\mu_3$$

$$\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3 = \frac{n^2}{(n-1)(n-2)}(\mu_2\mu_4 - \mu_3^2 - \mu_2^3)$$

The net result of this investigation of the application of balanced measures as presumptive values of moments in frequency distributions seems to be that, in view of the formal inconsistencies involved, it is necessary to carefully select the functions to which such measures are applied. The functions considered above are suggested as well adopted for this purpose and as probably sufficient for all practical purposes. If they are adopted as fundamental the resulting approximations for the Pearson constants $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ are

$$\bar{\beta}_1 = \bar{\mu}_3^2/\bar{\mu}_2^3 = \frac{n(n-1)}{(n-2)^2}\,\mu_3^2/\mu_2^3 = \frac{n(n-1)}{(n-2)^2}\,\beta_1 \qquad \text{and}$$

$$\bar{\beta}_2 - \bar{\beta}_1 - 1 = \frac{\bar{\mu}_2\bar{\mu}_4 - \bar{\mu}_3^2 - \bar{\mu}_2^3}{\bar{\mu}_2^3} = \frac{(n-1)^2}{n(n-2)}\;\frac{\mu_2\mu_4 - \mu_3^2 - \mu_2^3}{\mu_2^3}$$

$$= \frac{(n-1)^2}{n(n-2)}\;(\beta_2 - \beta_1 - 1).$$

It will be noted that the coefficient in the latter equation is very nearly unity for even moderate values of n.

A SHORT METHOD FOR SOLVING FOR A COEFFICIENT OF MULTIPLE CORRELATION

By

PAUL HORST

The method which we present presupposes a familiarity with the Doolittle method [1] for solving normal equations. We start with the determinant

$$(1) \qquad R = \begin{vmatrix} 1 & r_{12} & \cdots & r_{1n} \\ r_{12} & 1 & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{1n} & r_{2n} & & 1 \end{vmatrix}$$

where the elements are zero order coefficients of correlation.

Now the adjoint determinant of (1) may be written

$$(2) \qquad r = \begin{vmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{12} & R_{22} & \cdots & R_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ R_{1n} & R_{2n} & \cdots & R_{nn} \end{vmatrix}$$

where the elements are the cofactors of the elements in (1).

From the elementary theory of determinants [2] we know that

$$(3) \qquad r = R^{n-1}$$

The adjoint determinant of r may be designated by KR where

$$(4) \qquad KR = r^{n-1}$$

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[1] Mills, F. C., Statistical Methods, p. 577.

[2] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

From (3) and (4) we have

$$KR = R^{(n-1)^2}$$

or

$$K = R^{n(n-2)} \tag{5}$$

Hence the adjoint or r is obtained by multiplying each element of R by R^{n-2}. And if we write

$$KR = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{12} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{vmatrix} \tag{6}$$

then

$$\frac{A_{ij}}{R^{n-2}} = r_{ij} \tag{7}$$

so that (1) may be rewritten

$$R = \begin{vmatrix} \frac{A_{11}}{R^{n-2}} & \frac{A_{12}}{R^{n-2}} & \cdots & \frac{A_{1n}}{R^{n-2}} \\ \frac{A_{12}}{R^{n-2}} & \frac{A_{22}}{R^{n-2}} & \cdots & \frac{A_{2n}}{R^{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{A_{1n}}{R^{n-2}} & \frac{A_{2n}}{R^{n-2}} & \cdots & \frac{A_{nn}}{R^{n-2}} \end{vmatrix} \tag{8}$$

The numerators of the elements in (8) are the cofactors of the elements in (2).

Let us now consider (8) as the coefficients of a set of normal equations whose constant terms are zero, and let us follow through literally the Doolittle elimination process.

For simplicity we outline the reduction of a 4-variable problem as follows.

Reciprocal	1	2	3	4	α	β	γ	δ
$-\frac{R^2}{A_{11}}$	$\frac{A_{11}}{R^2}$ -1	$\frac{A_{12}}{R^2}$ $-\frac{A_{12}}{A_{11}}$	$\frac{A_{13}}{R^2}$ $-\frac{A_{13}}{A_{11}}$	$\frac{A_{14}}{R^2}$ $-\frac{A_{14}}{A_{11}}$	α_1		γ_1	δ
		$\frac{A_{22}}{R^2}$ $-\frac{A_{12}^2}{R^2A_{11}}$	$\frac{A_{23}}{R^2}$ $-\frac{A_{12}A_{13}}{R^2A_{11}}$	$\frac{A_{24}}{R^2}$ $-\frac{A_{12}A_{14}}{R^2A_{11}}$	α_2	β_{22}		
$-\frac{R^2A_{11}}{AA_{1122}}$		$\frac{AA_{1122}}{R^2A_{11}}$ -1	$\frac{AA_{1123}}{R^2A_{11}}$ $-\frac{A_{1123}}{A_{1122}}$	$\frac{AA_{1124}}{R^2A_{11}}$ $-\frac{A_{1124}}{A_{1122}}$			γ_2	δ_2
			$\frac{A_{33}}{R^2}$ $-\frac{A_{13}^2}{R^2A_{11}}$ $-\frac{AA_{1123}^2}{R^2A_{11}A_{1122}}$	$\frac{A_{34}}{R^2}$ $-\frac{A_{13}A_{14}}{R^2A_{11}}$ $-\frac{AA_{1123}A_{1124}}{R^2A_{11}A_{1122}}$	α_3	β_{23} β_{33}		
$\frac{R^2A_{1122}}{AA_{112233}}$			$\frac{AA_{112233}}{R^2A_{1122}}$ -1	$\frac{AA_{112234}}{R^2A_{1122}}$ $-\frac{A_{112234}}{A_{112233}}$			γ_3	δ_3
			$-\sum_2^n \beta_i n =$	$\frac{A'_{44}}{R^2}$ $-\frac{A_{14}^2}{R^2A_{11}}$ $-\frac{AA_{1124}^2}{R^2A_{11}A_{1122}}$ $-\frac{AA_{112234}^2}{R^2A_{1122}A_{112233}}$	α_4	β_{24} β_{34} β_{44}		
$\frac{R^2A_{112233}}{AA_{11223344}}$				$\frac{AA_{11223344}}{R^2A_{112233}}$ -1			γ_4	δ_4

The α-equations are the original equations with the coefficients to the left of the diagonal omitted. The β-equations are the product equations which are subtracted from the α-equations. The γ-equations are the reduced equations which may be represented symbolically by

$$\gamma = \alpha - \Sigma \beta \tag{9}$$

The δ-equations are the γ-equations divided by the negatives of their respective leading coefficients. That (9) is true may be readily proved from the theorem[8]

$$\begin{vmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{vmatrix} = AA_{ijkl} \tag{10}$$

Where the notation indicates cofactors rather than minors. The proof is made more obvious if (9) is written

$$\gamma = \left\{ \left[(\alpha - \beta_1) - \beta_2 \right] - \beta_3 \right\} - \beta_4 \quad etc.$$

for each successive subtraction reduces the determinant by one.

If we indicate the leading coefficients of the γ-equations by γ_{ii} we may prove that

$$R = \prod_1^n \gamma_{ii} \tag{11}$$

We have in the case of four variables

$$\prod_1^4 \gamma_{ii} = \frac{A_{11}}{R^2} \quad \frac{rA_{1122}}{R^2 A_{11}} \quad \frac{rA_{112233}}{R^2 A_{1122}} \quad \frac{rA_{11223344}}{R^2 A_{112233}} = \frac{r^3}{R^8} \tag{12}$$

$$A \equiv r$$

or from [8]

$$\prod_1^4 \gamma_{ii} = R$$

In the general case we have

$$\prod_1^n \gamma_{ii} = \frac{r^{n-1}}{R^{(n-2)n}} = \frac{R^{(n-1)^2}}{R^{n(n-2)}} = R$$

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[8] Bôcher, Maxime, Introduction to Higher Algebra, p. 33.

Now let us consider Kelley's equation for the coefficient of multiple correlation[4] with a slight change in notation to be consistent with the above,

$$(13) \qquad R_{n.12--(n-1)} = \sqrt{1-\frac{R}{R_{nn}}}$$

Obviously

$$(14) \qquad R_{nn} = \prod_{1}^{n-1} \gamma_{ii}$$

So that (13) becomes simply

$$(15) \qquad R_{n.12--(n-1)} = \sqrt{1-\gamma_{nn}}$$

But from (9) we have

$$(16) \qquad \gamma_{nn} = \alpha_{nn} - \sum_{2}^{n} \beta_{in}$$

hence $R_{n.12--(n-1)} = \sqrt{1-(\alpha_{nn}-\sum_{2}^{n}\beta_{in})}$

But $\alpha_{nn} = 1$ therefore we get

$$(17) \qquad R_{n.12--(n-1)} = \sqrt{\sum_{2}^{n} \beta_{in}}$$

In other words $R_{n.12--}$ is simply the square root of the last product summation.

From (17) it is obvious that the solution for the coefficient of multiple correlation is considerably shorter than the standard Doolittle solution for regression coefficients. All of the back solution work is eliminated, as is also the calculation of the last reciprocal.

The only caution needed with respect to the order of the variables is that the dependent variable shall be the nth variable.

The usual summation check method may be employed exactly as in the solution for regression coefficients.

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[4] Kelley, T. L., Statistical Method, p. 301, eq. 275.

A SHORT METHOD AND TABLES FOR THE CALCULATION OF THE AVERAGE AND STANDARD DEVIATION OF LOGARITHMIC DISTRIBUTIONS*

By

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In fitting various types of curves to reaction-time data,[1] the writer was impressed with the enormous amount of labor and boredom involved in the calculation of the constants of logarithmic distributions. Besides the constant use of a set of logarithm tables, it requires the tedium of squaring large numbers on a machine to compute the second moments of the distributions. In order to eliminate some of the labor involved in such a process, a short method was devised for the computation of the average and the standard deviation of logarithmic distributions.

The short method described in this paper was originally developed to facilitate the work of fitting logarithmic normal curves to a large number of reaction-time distributions, but dispersions approximating this type seem to be sufficiently common in economics and biology to warrant a more general use of short methods in the computation of the constants of such distributions. In the field of economics, logarithmic curves have been fitted with success to distributions of income and prices, and probably could be applied equally well to distributions of capital. Many skewed distributions can also be found in the fields of biology and psychology. Kapteyn fitted a logarithmic curve to a distribution of the minimum weights necessary to produce a sensation of pressure.[3] Kapteyn attempts to show that logically the normal curve is the exception and skew curves the rule. For example, if "the diameters of certain ripe berries" are distributed in a normal curve, their *volumes* will be distributed in an asymmetrical curve; in other words, volume increase will be dependent upon size, so that volume changes are greater for large berries than for small ones. That skew curves are the rule can be shown analytically. Suppose certain quantities z are distributed normally, and *any* other quantities x are expressed as functions of z, thus,

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*A portion of the work involved in this paper was carried out during the writer's tenure as a National Research Fellow.

[1] Cf. Facilitation and Inhibition. *Arch. Psychol.* No. 86, 56 p.

$$z = f(x)$$

Then,

(1) $$dz = f'(x)dx$$

If the frequency curve for the z's is,

(2) $$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{(z-M)^2}{2\sigma^2}}$$

then the frequency curve for the x's is,

(3) $$y \frac{N}{\sigma\sqrt{2\pi}} f'(x) e^{-\frac{[f(x)-M]^2}{2\sigma^2}}$$

It will be seen at once that the x's cannot be distributed normally provided x is a non-linear function of z. If we let $z = \log x$, then $dz = \frac{dx}{x}$ and equation (2) becomes

(4) $$y = \frac{N}{\sigma\sqrt{2\pi}} \frac{1}{x} e^{-\frac{(\log x - M)^2}{2\sigma^2}}$$

This is the logarithmic curve of distribution, the theory of which has been treated by several writers, one of the first and most important papers on this subject being that of McAllister.[4] The study of the properties of the logarithmic curve of error was undertaken by McAllister at the suggestion of Galton,[4] who saw the possibility of applying it to psychological and social phenomena.

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[3] J. C. Kapteyn, "Skew frequency curves in Biology and Statistics". Groningen. p. 42-43, 1903.

[4] The Law of the Geometric Mean. *Proc. Roy. Soc.* 29:367. (1879).

Dispersions approximating this type are illustrated by distributions which are definitely limited at the zero point, but a more definite presumption in favor of the logarithmic curve is indicated when the *real* origin, determined *a priori* or deduced from empirical considerations, does not correspond with the origin on the value scale.

Reaction-time distributions are good examples of dispersions where a displacement of the origin is indicated by empirical considerations. A little reflection will show that there must be a physiological limit for the speed of reaction. It takes a certain minimum time for the neuro-muscular machine to do its work. The time it takes for the machine to do its work constitutes an undisturbed region within which no deviations ever occur. Reaction-time dispersions approximate the logarithmic more closely than the normal curve of error. Investigations in the field of learning often give distributions which have origins other than the zero of the scale which can be determined *a priori* . . . .that is, the real origin follows *inevitably* from the conditions of the experiment. If the norm of mastery for learning a maze is two perfect trials out of three, then the criterion is such that an animal to learn a maze must make at least two perfect runs. In other words, the experimenter's criterion is such that no deviations could *possibly* occur under two trials.

In using the short method for finding the first and second moments of a logarithmic distribution, the computer must still resort to a table, but in this case it is only necessary to use a single page table instead of an extensive logarithm table. Furthermore, the labor of squaring the logarithms is eliminated. The short method can best be explained by following the process through an actual example.[5] This is illustrated in Table II on a distribution of reaction-times. Beginning at 70 (the real origin of the distribution is assumed to be at 70) the step-intervals are numbered from zero to the end of the distribution. Under the log x column of Table I the value for each step is found and multiplied by the frequency for each step. This operation gives the values shown in the $F \log x$ column of Table II. The sum of these values divided by N (number of cases) gives the correction C. The average ($\log G_1$) for the logarithmic distribution is finally found by adding a factor K to the correction C. The constant K depends upon the length of the step-interval. In this distribution, the length of the step-interval is ten. Looking under column K of Table I, we find that the value of K for a step-interval of ten units is equal to .69897. The geometric mean (G) of the distribution is found by adding 70 to (G_1).[6]

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[4] The Geometric Mean in Vital and Social Statistics. *Proc. Roy. Soc.* 29:365. (1879).

The process of finding the second moment and standard deviation (σ_g) is similar to that for finding the first moment and the average. In one respect it is simpler: no correction has to be added for the length of step. The $F \log^2 x$ column is obtained by multiplying the value for each step in the $\log^2 x$ column of Table I by its appropriate frequency. The sum of these divided by N (number of cases) gives the crude unit moment. The square of the correction C is then subtracted to give the corrected unit moment around the average. The square root of the corrected unit moment around the average gives the standard deviation (σ_g), and the antilog of σ_g gives the standard deviation ratio (σ_r).

The formula for finding the average of a logarithmic distribution is,

$$\log G_1 = \frac{\Sigma F \log x}{N} + K = C + K$$

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[5] For those interested, the proof of the formulae for getting the average and standard deviation is given in an appendix at the end of this paper.

[6] One would expect the geometric mean to be different if the origin were taken at a point other than 70. An origin at 70 was assumed because it results in an extremely good fit to the distribution. In this case, 70 would correspond to the physiological limit below which no deviations ever occur.

TABLE II

Time	*F*	Step	*F* log *x*	*F* $\log^2 x$
70	1	1		
80	3	2	1.431363	.682934
90	14	3	9.785580	6.839826
100	40	4	33.803921	28.567627
10	55	5	52.483338	50.081832
20	60	6	62.483561	65.069923
30	52	7	57.925054	64.525229
40	46	8	54.100197	63.626769
50	33	9	40.604814	49.962150
60	28	10	35.805100	45.785901
70	21	11	27.766605	36.713541
80	14	12	19.064189	25.960237
90	9	13	12.581460	17.588126
200	7	14	10.019546	14.341615
10	7	15	10.236785	14.970255
20	4	16	5.965446	8.896638
30	3	17	4.555541	6.917653
40	1	18	1.544068	2.384146
50				
60	1	20	1.591064	2.531486
70				
80	1	22	1.633468	2.668219
	400		400)443.381100	400)508.114107
				1.270285
			C = 1.108452	$C^2 =$ 1.228667
			K = .698970	
				$\sigma_g^2 =$ .041618
			log $G_1 =$ 1.807422	
				$\sigma_g =$ .204004
For origin at 70			$G_1 =$ 65.8	
			$G =$ 135.8	$\sigma_r =$ 1.59

TABLE I

Step	Log x	$(\text{Log } x)^2$	Step Interval	K
1	.00000 00000	.00000 00000	1	.30102 99956
2	.47712 12547	.22764 46917	2	.00000 00000
3	.69897 00043	.48855 90669	3	.17609 12590
4	.84509 80400	.71419 06972	4	.30102 99956
5	.95424 25094	.91057 87668	5	.39794 00086
6	1.04139 26851	1.08449 87247	6	.47712 12547
7	1.11394 33523	1.24086 97921	7	.54406 80443
8	1.17609 12590	1.38319 06496	8	.60205 99913
9	1.23044 89213	1.51400 45481	9	.64321 25137
10	1.27875 36009	1.63521 07719	10	.69897 00043
11	1.32221 92947	1.74826 38633	11	.74036 26894
12	1.36172 78360	1.85430 26993	12	.77815 12503
13	1.39794 00086	1.95423 62678	13	.81291 33566
14	1.43136 37641	2.04880 22253	14	.84509 80400
15	1.46239 79978	2.13860 79042	15	.87506 12633
16	1.49136 16938	2.22415 97018	16	.90308 99869
17	1.51851 39398	2.30588 45856	17	.92941 89257
18	1.54406 80443	2.38414 61255	18	.95424 25094
19	1.56820 17240	2.45925 66473	19	.97772 36052
20	1.59106 46070	2.53148 65837	20	1.00000 00000
21	1.61278 38567	2.60107 17684	21	1.02118 92990
22	1.63346 84555	2.66821 91953	22	1.04139 26851
23	1.65321 25137	2.73311 16157	23	1.06069 78403
24	1.67209 78579	2.79591 12465	24	1.07918 12460
25	1.69019 60800	2.85676 27889	25	1.09691 00130
26	1.70757 01760	2.91579 59062	26	1.11394 33523
27	1.72427 58696	2.97312 72744	27	1.13033 37684
28	1.74036 26894	3.02886 22909	28	1.14612 80356
29	1.75587 48556	3.08309 65087	29	1.16136 80022
30	1.77085 20116	3.13591 68471	30	1.17609 12590
31	1.78532 98350	3.18740 26197	31	1.19033 16981
32	1.79934 05494	3.23762 64129	32	1.20411 99826
33	1.81291 33566	3.28665 48386	33	1.21748 39442
34	1.82607 48027	3.33454 91850	34	1.23044 89213
35	1.83884 90907	3.38136 59785	35	1.24303 80486
36	1.85125 83487	3.42715 74737	36	1.25527 25051

37	1.86332 28601	3.47197 20810	37	1.26717 17284
38	1.87506 12633	3.51585 47414	38	1.27875 36009
39	1.88649 07251	3.55884 72561	39	1.29003 46113
40	1.89762 70912	3.60098 85775	40	1.30102 99956
41	1.90848 50188	3.64231 50672	41	1.31175 38610
42	1.91907 80923	3.68286 07246	42	1.32221 92947
43	1.92941 89257	3.72265 73909	43	1.33243 84599
44	1.93951 92526	3.76173 49312	44	1.34242 26808
45	1.94939 00066	3.80012 13980	45	1.35218 25181
46	1.95904 13923	3.83784 31768	46	1.36172 78360
47	1.96848 29485	3.87492 51187	47	1.37106 78622
48	1.97772 36052	3.91139 06589	48	1.38021 12417
49	1.98677 17342	3.94726 19240	49	1.38916 60843
50	1.99563 51945	3.98255 98299	50	1.39794 00086

and, $G_l = \text{antilog}(C + K)$

where G_l is the geometric mean measured from an origin which may be other than the zero of the scale. The geometric mean (G) measured from the zero of the value scale is,

$$G = G_l + (\text{displacement of the origin})$$

The formula for finding the standard deviation around the average is,

$$\sigma_g = \sqrt{\frac{\Sigma F \log^2 x}{N}} - C^2$$

and, $\sigma_r = \text{antilog}\ \sigma_g$

Summary of steps in the calculation of the average ($\log G_l$) by the short method:

1. Beginning at the origin, find the deviation of the mid-point of each step-interval from the origin in units of step-interval.
2. Using Table I, find the log x of each step-deviation and weight it by its appropriate F (frequency).

3. Find the *sum* of the $F \log^2 x$'s, and divide this sum by N (number of cases). This gives the *correction* C.
4. Using Table I, find the value of K corresponding to the number of units in the step-interval. Add the factor K to the correction C to get the average ($\log G_1$).

Summary of steps in the calculation of the standard deviation (σ_g) around the average ($\log G_1$) by the short method:

1. Using Table 1, find the $\log^2 x$ of each step-deviation and weight it by its appropriate frequency.
2. Find the *sum* of the $F \log^2 x$'s; and divide this sum by N.
3. Then subtract the square of the correction C to get the *second unit moment* around the average.
4. Extract the square root of the second unit moment to obtain the standard deviation (σ_g).

APPENDIX

DEDUCTION OF THE FORMULAE FOR THE SHORT METHOD

Let $\log G$ be the logarithmic mean, x, the length of step, $f_1 \ldots\ldots f_n$ the frequencies for successive steps, and $m_1, m_2 \ldots\ldots m_n$ the mid-points of the steps for origin at zero. Then the first mid-point, m_1, is at $x/2$ the second, m_2, is at $3x/2$ etc. For convenience, these items may be arranged in the form of a table.

Midpoint (M)	F	F log M	F log² M
m_1	f_1	$f_1 \log x/2$	$f_1 \log^2 x/2$
m_2	f_2	$f_2 \log 3x/2$	$f_2 \log$
$\vdots$	$\vdots$	$\vdots$	
m_n	f_n	$f_n \log \frac{2n-1}{2} x$	$f_n \log^2 \frac{2n-1}{2} x$
	N		

$$\log G = \frac{\Sigma f_n \log \frac{2n-1}{2} x}{N}, \quad \sigma_g^2 = \frac{\Sigma f_n \log^2 \frac{2n-1}{2} x}{N} - \log^2 G$$

where G is the geometric mean, and σ_g is the standard deviation around $\log G$.

We have, therefore,

$$\log G = \frac{f_1 \log \frac{x}{2} + f_1 \log \frac{3x}{2} + \cdots\cdots + f_n \log \left(\frac{2n-1}{2}\right) x}{N}$$

Stating the logarithm of each fraction as the sum or difference of the logarithms of its factors, we have,

$$\log G = \frac{f_1 (\log 1 - \log 2 + \log x) + \cdots + f_n \left[\log(2n-1) - \log 2 + \log x\right]}{N}$$

$$= \frac{f_1 \log 1 + f_2 \log 3 + \cdots + f_n \log(2n-1)}{N} + \log x - \log 2$$

$$= \frac{\Sigma \left[f_n \log(2n-1)\right]}{N} + \log x - \log 2$$

Since $$\log x = \frac{\Sigma f \log x}{N}$$

and, $$\log 2 = \frac{\Sigma f \log 2}{N}$$

Letting $$K = \log x - \log 2$$

and, $$C = \frac{\Sigma \left[f_n \log(2n-1)\right]}{N}$$

we finally have,

$$\log G = C + K \quad .$$

Where C is the correction, and K is the constant indicated in Table I.

For the second unit moment around $\log G$, we have,

$$\sigma_g^2 = \frac{\Sigma\left[f_n \log^2 \left(\frac{2n-1}{2}\right) x\right]}{N} - \left(\frac{\Sigma\left[f_n \log\left(\frac{2n-1}{2}\right) x\right]}{N}\right)^2$$

$$= \frac{\Sigma\left\{f_n\left[\log^2\left(\frac{2n-1}{2}\right) + 2\log\left(\frac{2n-1}{2}\right)\log x + \log^2 x\right]\right\}}{N}$$

$$- \left\{\left(\frac{\Sigma f_n \log\left(\frac{2n-1}{2}\right)}{N}\right)^2 + \frac{2\Sigma\left[f_n \log\left(\frac{2n-1}{2}\right)\right]\log x}{N} + \log^2 x\right\}$$

Expanding again and collecting terms, we have,

$$\sigma_g^2 = \frac{\Sigma\left\{f_n\left[\log(2n-1)\right]^2\right\}}{N} - \left\{\frac{\Sigma\left[f_n \log(2n-1)\right]}{N}\right\}^2$$

$$= \frac{\Sigma\left\{f_n\left[\log(2n-1)\right]^2\right\}}{N} - C^2$$

In Table I, $x = (2n-1)$ so that,

$$\log x = \log(2n-1)$$

and $(\log x)^2 = \left[\log(2n-1)\right]^2$

For each step, 1, 2, 3 . . . . n, the corresponding values of x are 1, 3, 5 . . . . $2n-1$. Note that x as used in the table is not the same as x as used in the deduction of the formulae.

The figures in Table I are accurate to ten places of decimals.

The log x column consists simply of the logarithms of odd numbers from one to one hundred. K was computed by subtracting log 2 from the logarithm of each number indicated in the "step interval" column. The $(\log x)^2$ column was computed by squaring fifteen place logarithms with the aid of calculating machines. This had to be done by indirect methods through the use of the simple algebraic relationship, $(a+r)^2 = a^2+2ar+r^2$, where a is the first part of the number and is the remainder. The table was computed by two different persons and checked on two different calculating machines by each person.

ON SYMMETRIC FUNCTIONS OF MORE THAN ONE VARIABLE AND OF FREQUENCY FUNCTIONS

By

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In a paper published in this journal[1] the writer has developed a simple differential operator method for expressing any symmetric function of the n variates $x_1, x_2, \cdots\cdots, x_n$ as a rational, integral, algebraic function of the power sums $s_1, s_2, \cdots, s_w$ where w is the weight of the symmetric function and

$$s_k = \sum x_i^k = x_1^k + x_2^k + \cdots\cdots + x_n^k.$$

The transformation to moments is then simply a matter of recognizing that $n\mu'_k = s_k$ if μ'_k is the kth moment of the n variates with respect to the origin from which they are measured. If the origin is at the arithmetic mean of the n variates the prime may be dropped and then $n\mu_k = s_k$.

In the above mentioned paper the variates x_i are of the serial distribution type, but, of course, not necessarily integers. The extensions to the case of more than one set of variates and to frequency functions now suggest themselves. It is the purpose of this note to discuss these problems simultaneously.

Suppose that two sets, of n variates each, $x_1, x_2, \cdots; x_n$ and $y_1, y_2, \cdots, y_n$ are given and that x_i, y_i $(i = 1, 2, \ldots\ldots, n)$ are corresponding pairs. Modifying the partition notation used in the previous paper the symmetric function to be considered may be written in the form $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots\cdots ; b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$ i.e. the sum of all such terms as

$$x_1^{a_1} x_2^{a_1} \cdots x_{n_1}^{a_1} x_{n_1+1}^{a_2} \cdots x_{n_1+n_2}^{a_2} \cdots y_1^{b_1} y_2^{b_1} \cdots y_{m_1}^{b_1} y_{m_1+1}^{b_2} \cdots y_{m_1+m_2}^{b_2} \cdots$$

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[1] Symmetric Functions and Symmetric Functions of Symmetric Functions, Vol. II. No. 2 (May, 1931), pp. 102-149.

where

$$a_1, a_2, a_3, \cdots, b_1, b_2, b_3, \cdots n_1, n_2, n_3, \cdots m_1, m_2, m_3, \cdots$$

are positive integers and where

$$a_1 > a_2 > a_3 > \cdots > 0.$$

$$n_1 + n_2 + n_3 + \cdots = m_1 + m_2 + m_3 + \cdots.$$

e.g. $(3^2 2.451) = (332.451) = \sum x_i^3 x_j^3 x_k^2 y_i^4 y_j^5 y_k$, $i \neq j \neq k$. Obviously the order of the parts in the bipartite notation will be important, corresponding parts being equidistant from the left of their respective sections of the partition. The double partition will be said to be of weight $w_1 = a_1 n_1 + a_2 n_2 + a_3 n_3 + \cdots$ in x and of weight $w_2 = b_1 m_1 + b_2 m_2 + b_3 m_3 + \cdots$ in y. There will be no loss of generality if it is assumed that $w_1 \geq w_2$.

By a procedure similar to that employed in the first chapter of the paper already referred to, it may be shown that any symmetric function of the type defined above can be expressed as rational, integral, algebraic function of the symmetric functions

$$s_{11}, \; s_{12}, \; s_{21}, \cdots\cdots, s_{w_1 w_2}$$

where

$$s_{hk} = \sum_{i=1}^{n} x_i^h y_i^k, \quad h = 1, 2, \cdots, w_1, \quad k = 1, 2, \cdots, w_2.$$

Moreover, the terms of this function will be isobaric of weight w_1 in x, isobaric of weight w_2 in y, and hence isobaric of weight $w_1 + w_2$ in x and y together.

e. g. Multiply s_{21} by itself:

$$s_{21}^2 = (x_1^2 y_1 + x_2^2 y_2 + \cdots + x_n^2 y_n)^2$$

$$= (x_1^4 y_1^2 + x_2^4 y_2^2 + \cdots + x_n^4 y_n^2) + 2(x_1^2 x_2^2 y_1 y_2 + x_1^2 x_3^2 y_1 y_3 + \cdots)$$

$$= s_{42} + 2(2^2 \cdot 1^2).$$

$$(2^2 \cdot 1^2) = (s_{21}^2 - s_{42})/2,$$

each term being of weight 4 in x, 2 in y, and 6 in x and y together. It is possible then to write

$$(a_1^{n_1}a_2^{n_2}a_3^{n_3}\dots b_1^{m_1}b_2^{m_2}b_3^{m_3}\dots) \equiv$$

$$f(s_{11}, s_{12}, \dots, s_{ij}, \dots, s_{w_1w_2}) \equiv f$$

where f stands for a rational, integral, algebraic function of the sums (or product moments) s_{ij}, $i=1,\dots,w_1$, $j=1,\dots,w_2$, isobaric as explained above. Suppose that a new pair of variates $x_{n+1}=\alpha$, $y_{n+1}=\beta$ are introduced. Obviously s_{ij} becomes $s_{ij} + \alpha^i\beta^j$. Hence applying Taylor's Theorem f becomes

$$f + (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \dots + \alpha^i\beta^j d_{ij} + \dots + \alpha^{w_1}\beta^{w_2} d_{w_1w_2}) f$$

$$+ (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \dots + \alpha^i\beta^j d_{ij} + \dots + \alpha^{w_1}\beta^{w_2} d_{w_1w_2})^2 f/2!$$

$$+ \dots\dots\dots$$

$$+ (\alpha\beta d_{11} + \alpha\beta^2 d_{12} + \dots + \alpha^i\beta^j d_{ij} + \dots + \alpha^{w_1}\beta^{w_2} d_{w_1w_2})^{w_2} f/w_2!$$

all other terms being identically zero, where $d_{ij} = \partial/\partial_{ij}$ and $d_{ij}^k = \partial^k/\partial s_{ij}^k$, $i=1,\dots,w_1$, $j=1,\dots,w_2$.

Using the multinomial theorem and collecting coefficients of $\alpha^i\beta^j$ the above expression may be written in the form

$$(1 + \alpha\beta D_{11} + \alpha\beta^2 D_{12} + \dots + \alpha^i\beta^j D_{ij} + \dots) f$$

where

$$(1) \quad \left[\begin{array}{l} \left[\, D_{11} = d_{11}, \right. \\ \left[\begin{array}{l} D_{12} = d_{12}, \\ D_{21} = d_{21} \end{array}\right. \\ \left[\begin{array}{l} D_{13} = d_{13} \\ D_{22} = d_{22} + d_{11}^2/2, \\ D_{31} = d_{31}, \end{array}\right. \\ \left[\begin{array}{l} D_{14} = d_{14}, \\ D_{23} = d_{23} + d_{11} d_{12}, \\ D_{32} = d_{32} + d_{11} d_{21}, \\ D_{41} = d_{41}, \end{array}\right. \end{array}\right.$$

$$\left\{\begin{array}{l} D_{15} = d_{15}, \\ D_{24} = d_{24} + d_{11} d_{13} + d_{12}^2/2, \\ D_{33} = d_{33} + d_{11} d_{22} + d_{12} d_{21} + d_{11}^3/6, \\ D_{42} = d_{42} + d_{11} d_{31} + d_{21}^2/2, \\ D_{51} = d_{51} \end{array}\right.$$

$$D_{ij} = \sum \frac{d_{i_1 j_1}^{k_1} \, d_{i_2 j_2}^{k_2} \, d_{i_3 j_3}^{k_3} \cdots\cdots}{k_1! \, k_2! \, k_3! \cdots\cdots}, \quad i=1,\cdots,w_1,\ j=1,\ \cdots,w_2,$$

where
$$k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots\cdots\cdots = i,$$
$$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots\cdots = j,$$

$i_1, i_2, i_3, \cdots\cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers. The inverse relation is given by

$$d_{ij} = \sum \frac{(-1)^{v+1}(v-1)!\, D_{i_1 j_1}^{k_1} \, D_{i_2 j_2}^{k_2} \, D_{i_3 j_3}^{k_3} \cdots\cdots}{k_1! \quad k_2! \quad k_3! \cdots\cdots},$$

where
$$i=1, \cdots\cdots, w_1, \quad j=1, \cdots\cdots, w_2,$$
$$k_1 i_1 + k_2 i_2 + k_3 i_3 + \cdots\cdots = i,$$
$$k_1 j_1 + k_2 j_2 + k_3 j_3 + \cdots\cdots = j,$$
$$k_1 + k_2 + k_3 + \cdots\cdots\cdots\cdots = v,$$

$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots$ being positive integers.

The effect of the new variates α, β on

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots\cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

is to replace this symmetric function by

$$(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots \quad b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$
$$+ \alpha^{a_1} \beta^{b_1} (a_1^{n_1-1} a_2^{n_2} a_3^{n_3} \cdots\cdots b_1^{m_1-1} b_2^{m_2} b_3^{m_3} \cdots)$$
$$+ \alpha^{a_2} \beta^{b_2} (a_1^{n_1} a_2^{n_2-1} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2-1} b_3^{m_3} \cdots) + \cdots,$$

Hence replacing f by $(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$, then

$$(1 + \alpha\beta D_{11} + \alpha\beta^2 D_{12} + \alpha^2\beta D_{21} + \cdots$$
$$+ \alpha^i \beta^j D_{ij} + \cdots)(a_1^{n_1} a_2^{n_2} a_3^{n_3} \cdots b_1^{m_1} b_2^{m_2} b_3^{m_3} \cdots)$$

$$=(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)$$

$$+\alpha^{a_1}\beta^{b_1}(a_1^{n_1-1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1-1}b_2^{m_2}b_3^{m_3}\cdots)$$

$$+\alpha^{a_2}\beta^{b_2}(a_1^{n_1}a_2^{n_2-1}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2-1}b_3^{m_3}\cdots)+\cdots.$$

Equating coefficients of like terms in α and β it is seen that

$$(2)\quad\left[\begin{array}{l} D_{a_1b_1}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots) \\ =(a_1^{n_1-1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1-1}b_2^{m_2}b_3^{m_3}\cdots), \\ D_{a_2b_2}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots) \\ =(a_1^{n_1}a_2^{n_2-1}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2-1}b_3^{m_3}\cdots), \\ \text{etc. and} \\ D_{hk}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)=0 \qquad \text{if both} \\ h \text{ and } k \text{ are not among } a_1, a_2, a_3, \cdots \text{ and } b_1, b_2, b_3, \cdots \\ \text{respectively. Hence also} \\ D^{t_1}_{a_1b_1}D^{t_2}_{a_2b_2}D^{t_3}_{a_3b_3}\cdots D^{t_k}_{a_kb_k}(a_1^{n_1}a_2^{n_2}a_3^{n_3}\cdots b_1^{m_1}b_2^{m_2}b_3^{m_3}\cdots)=0 \end{array}\right.$$

if $t_1, t_2, t_3, \cdots, t_k$ are not all respectively less than or equal to $n_1, n_2, n_3, \cdots, n_k$ and also respectively less than or equal to $m_1, m_2, m_3, \cdots, m_k$ where $t_1, t_2, t_3, \cdots, t_k$ are positive integers or zeros; that is, if even one of the values of t is greater than the corresponding n or m then the effect of the operation is to give zero. This last property of the D operators is important not only in minimizing the work of computation as will be seen in the illustrations given below, but will be of fundamental importance in the theorem to be stated in the closing paragraph of this paper. It should be noted here that the multiplication of the operators is commutative.

To illustrate the application of the operators consider:
a) $\Gamma(21.1^2)=c_1s_{21}s_{11}+c_2s_{32}$ where c_1 and c_2 are constants to be determined. These are the only terms which satisfy the weight conditions. Operating on the left with D_{21} and on the right with

d_{21} gives $(1.1) = c_1 s_{11}$, i.e. $s_{11} = c_1 s_{11}$, hence $c_1 = 1$. Operating on the left with D_{32} and on the right with $d_{32} + d_{11} d_{21}$ gives $0 = c_1 + c_2$, hence $c_2 = -1$ and thus $(21.1^2) = s_{21} s_{11} - s_{32}$.

b) $(321.1^3) = c_1 s_{63} + c_2 s_{52} s_{11} + c_3 s_{51} s_{12} + c_4 s_{42} s_{21} + c_5 s_{41} s_{22}$

$$+ c_6 s_{41} s_{11}^2 + c_7 s_{32} s_{31} + c_8 s_{31} s_{21} s_{11} + c_9 s_{21}^3 .$$

These are the only terms which satisfy the weight conditions. If the expressions for all the symmetric functions of lower weights are known then to determine the constants it is sufficient to operate on the left with

$D_{11}, D_{21}, D_{31}, D_{41}, D_{51}, D_{63}$ and on the right with their respective equivalents in terms of d. However if the expressions for the symmetric functions of lower weights are not known then it is perhaps simpler to operate with

$d_{12}, d_{11}^2, d_{21}^2, \quad d_{31} d_{21} d_{11}, \quad d_{31} d_{32}, d_{41} d_{22}, d_{42} d_{21},$

$d_{11} d_{52}, d_{63}$ on the right and with their respective equivalents in terms of D on the left. d_{12} and D_{12} give $0 = c_3 s_{51}$ and hence $c_3 = 0$. d_{11}^2 and D_{11}^2 give $0 = 2 c_6 s_{41}$ and $c_6 = 0$. d_{21}^2 and D_{21}^2 give $0 = 6 c_9 s_{21}$ and $c_9 = 0$. Similarly $d_{31} d_{21} d_{11}$ and $D_{31} D_{21} D_{11}$ give $c_8 = 1$. $d_{31} d_{32}$ and $D_{31} (D_{32} - D_{21} D_{11})$ give $c_7 = -1$. $d_{41} d_{22}$ and $D_{41} (D_{22} - D_{11}^2/2)$ give $c_5 = 0$. $d_{21} d_{42}$ and $D_{21} (D_{42} - D_{31} D_{11} - D_{21}^2/2)$ give $C_4 = -1$. $d_{11} d_{52}$ and $D_{11} (D_{52} - D_{41} D_{11} - D_{31} D_{21})$ give $c_2 = -1$:

d_{63} and $D_{63} - D_{52} D_{11} - D_{51} D_{12} - D_{42} D_{21} - D_{41} D_{22} + D_{41} D_{11}^2 - D_{32} D_{31}$ $+ 2 D_{31} D_{21} D_{11} + D_{21}^3/2$ give $c_1 = 2$. Hence $(321.1^3) = 2 s_{63} -$ $s_{52} s_{11} - s_{42} s_{21} - s_{32} s_{31} + s_{31} s_{21} s_{11}$.

Suppose that $y_k = 1$, $k = 1, 2, \cdots, n$. Then s_{ij} is simply s_i and D_{ij}, d_{ij} have no meaning except when $i = j$ and then D_{ii} and d_{ii} become the operators D_i and d_i respectively, of the earlier paper.

The operator relations for any number of sets of corresponding variates are now obvious. For instance, in the case of 3 sets

x_i, y_i, z_i the result is

$$D_{ijk} = \sum \frac{d_{i_1j_1k_1}^{h_1} \, d_{i_2j_2k_2}^{h_2} \, d_{i_3j_3k_3}^{h_3} \cdots\cdots}{h_1! \quad h_2! \quad h_3! \cdots\cdots},$$

$$i = 1, 2, \cdots\cdots, w_1,$$

$$j = 1, 2, \cdots\cdots, w_2,$$

$$k = 1, 2, \cdots\cdots, w_3,$$

$$h_1 i_1 + h_2 i_2 + h_3 i_3 + \cdots\cdots = i,$$

$$h_1 j_1 + h_2 j_2 + h_3 j_3 + \cdots\cdots = j,$$

$$h_1 k_1 + h_2 k_2 + h_3 k_3 + \cdots\cdots = k,$$

$$i_1, i_2, i_3, \cdots, j_1, j_2, j_3, \cdots, k_1, k_2, k_3, \cdots, h_1, h_2, h_3, \cdots$$

being positive integers, w_1, w_2, w_3 the weights in xyz respectively of the symmetric function,

$$d_{ijk} = \partial / \partial s_{ijk}, \quad s_{ijk} = \sum_{t=1}^{n} x_t^i y_t^j z_t^k .$$

Returning now to the case of two variates x and y, suppose that x_i takes on only integral values for $i = 1, 2, \ldots, n$ and that $y_i = f(x_i)$ where $f(x_i)$ is thought of as the frequency corresponding to $x = x_i$ If, further, $b_1 = b_2 = b_3 = \cdots\cdots = 1$ then the operators developed in this note give the expressions for $(a^\alpha b^\beta c^\gamma \ldots)$ of the earlier paper when each serial x_i^k is replaced by $x_i^k f(x_i)$. More generally, let $y_i = f(x_i) \Delta x_i$ represent the frequency of x in the interval Δx_i. If x takes on only integral values then of course $\Delta x_i = 1$, $i = 1, 2, \ldots\ldots, n$. Consider

$$(32.11) = \sum_{\substack{i=1 \\ j=1}}^{n} x_i^3 x_j^2 f(x_i) f(x_j) \Delta x_i \, \Delta x_j , \quad i \neq j ,$$

$$= \sum_{i=1}^{n} x_i^3 f(x_i) \Delta x_i . \sum_{j=1}^{n} x_j^2 f(x_j) \Delta x_j - \sum_{i=1}^{n} x_i^5 f^2(x_i) \overline{\Delta x}_i^2 .$$

If the lower and upper bounds for x are respectively a and b then in the limit as n becomes infinite and the maximum Δx_i, $i = 1, 2, \cdots, n$, approaches zero, the last summation on the right approaches zero, $f(x)$ being an ordinary frequency function. Thus in this limiting

$$(32.11) = \int_a^b\int_a^b x_i^3 x_j^2 f(x_i) f(x_j)\,dx_i\,dx_j,\ i \neq j,$$

$$= \int_a^b x_i^3 f(x_i)\,dx_i \cdot \int_a^b x_j^2 f(x_j)\,dx_j.$$

In general, under these limiting conditions, any summation $\sum_{i=1}^{n} x_i^k f^r(x_i)\overline{\Delta x_i}^{\,r}$ approaches zero, if r is greater than 1, k and r being positive integers. For let $\mathcal{E}_n$ be the maximum Δx_i for specified n. Now $x_i^k f^r(x_i) \leqq M$ for $i = 1, 2, \ldots, n$. Hence

$$\sum_{i=1}^{n} x_i^k f^r(x_i)\overline{\Delta x_i}^{\,r} \leqq M\sum_{i=1}^{n}\overline{\Delta x_i}^{\,r} \leqq M\mathcal{E}_n^{r-1}\sum_{i=1}^{n}\Delta x_i = M\mathcal{E}_n^{r-1}(b-a)$$

which approaches zero with $\mathcal{E}_n$. This establishes the well known statement that, the values of x being independent,

$$\int_a^b\int_a^b\cdots\int_a^b x_{i_1}^{a_1} x_{i_2}^{a_2}\cdots x_{i_k}^{a_k} f(x_{i_1}) f(x_{i_2})\cdots f(x_{i_k})\,dx_{i_1}\,dx_{i_2}\cdots dx_{i_k}$$

$$= \int_a^b x_{i_1}^{a_1} f(x_{i_1})\,dx_{i_1} \cdot \int_a^b x_{i_2}^{a_2} f(x_{i_2})\,dx_{i_2} \cdots \int_a^b x_{i_k}^{a_k} f(x_{i_k})\,dx_{i_k}.$$

For, under the above limiting conditions, all those terms which contain a sum $s_{hk} = \sum x_i^h f^k(x_i)\overline{\Delta x_i}^{\,k}$ with k greater than 1 must vanish. By the last property of the D operators given in (2) it is seen that there is always only one term which does not contain such an s_{hk}; and from this term arises the product of the definite integrals.

A GENERALIZED ERROR FUNCTION*

By

ALBERT WERTHEIMER

I. INTRODUCTION

Given a set of observed values ℓ_i ($i = 1, 2, 3, \ldots, n$.) obtained from n observations assumed to be made on the same quantity, ℓ, under the same conditions. We seek to determine two functions $f(P, \ell_i)$ and $\phi(P, \ell_i)$ such that

$$f(P, \ell_i) = 0, \qquad (i = 1, 2, 3, \ldots, n)$$

defines p as a unique value assigned to the observed quantity; and $\phi(P, \ell_i)d\ell_i$ gives to within infinitesimals of higher order the probability that if another observation is made, the observed value will lie in the interval

$$\ell_i \leqq \ell \leqq \ell_i + d\ell_i .$$

Gauss determined the ϕ function to be the so-called normal error law namely,

$$\phi(P, \ell_i) = ce^{-h^2(P-\ell_i)^2}$$

on the basis of the following assumptions,

(a) The product $\prod_i \phi(P, \ell_i)$ is to be a maximum with respect to p. Thus

$$\sum_i \frac{\partial}{\partial p} \log \phi(P, \ell_i) = 0,$$

$$\sum_i \frac{\partial^2}{\partial p^2} \log \phi(P, \ell_i) \not> 0.$$

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*Presented to the American Mathematical Society, December 28, 1931.

(b) The unique value p is the arithmetic mean of the observations. Thus

$$f(P, \ell_i) = \sum_i (P - \ell_i).$$

(c) The probability function is a function of $(P - \ell_i)$. Thus

$$\phi(P, \ell_i) \equiv \phi(P - \ell_i).$$

Poincaré[1] on the basis of the first two assumptions only obtained the error function

$$\phi(P, \ell_i) = \theta(\ell_i) e^{W(P) + \ell_i V(P)},$$

where

$$\frac{dW}{dP} + p \frac{dV}{dP} = 0.$$

In this paper we assume the unique value p to be defined by a function satisfying certain conditions, and obtain on the basis of assumption (a) a more general error function from which the so-called normal error law, the Poincare function, and other forms of the error function as well as the Pearson curves are obtained as special cases.

2. *The unique value p.*

We now make the following assumptions:

I: The unique value p is defined explicitly as a function of the observed values in the region $a \leq \ell_i \leq b$. Thus

$$P - F(\ell_1, \ell_2, \ell_3, \cdots, \ell_n) = 0,$$

where F is single valued, continuous with continuous derivatives up to the second order.

II: The value of p is independent of the order in which the observations are obtained. Thus F is a symmetric function.

III: The change in p due to a change in one of the observed values, say ℓ_i, is a function of p and ℓ_i only. Thus

$$F_{\ell_i} = F_{\ell_i}(F, \ell_i).$$

---

[1] H. Poincare, *Calcul des Probabilites* (1912), p. 171.

IV: If p is regarded as a function of a single variable, say l_i, while all the others are regarded as constants, then with respect to this variable p is a monotonic function and is not constant in any portion of the interval in which it is defined. Thus

$$F_{l_i} \neq 0$$

for all i's.

We have then for the determination of the ϕ function the two equations

$$\sum_i \frac{\partial}{\partial P} \log \phi(P, l_i) = 0, \tag{1}$$

$$P - F(l_1, l_2, l_3, \cdots, l_n) = 0, \tag{2}$$

which must be simultaneously satisfied for any set of values in the region defined.

3. *The g function.*

We will now show by means of the following theorems that if F satisfies the given conditions, then there exists a unique function $g(P, l_i)$ such that the equation

$$\sum_i g(P, l_i) = 0,$$

is identical with equation (2).

THEOREM I.

Given a function of n variables,

$$F(x_1', x_2^2, x_3^3, \cdots, x^n),$$

continuous with continuous non-vanishing first derivatives in the region defined, such that,

$$F_{x^i} = \psi^i(F, x^i);$$

then $\psi^i(F, x^i)$ must be in the form of a product of a function of F and a function of t_i. Thus

$$\psi^i(F, x^i) = \omega(F)\beta^i(x^i).$$

Proof:

We have

$$F_{x^i x^j} = \psi_F^i(F, x^i)\psi^j(F, x^j),$$

and

$$F_{x^j x^i} = \psi_F^j(F, x^j)\psi^i(F, x^i).$$

Hence:

$$\frac{\psi_F^i(F, x^i)}{\psi^i(F, x^i)} = \frac{\psi_F^j(F, x^j)}{\psi^j(F, x^j)} = \dots = \frac{\psi_F^n(F, x^n)}{\psi^n(F, x^n)} = \eta(F).$$

Integrating, we get

$$\log \psi^i(F, x^i) = \int \eta(F)\,dF + \xi^i(x^i).$$

from which it follows that

$$\psi^i(F, x^i) = \omega(F)\beta^i(x^i).$$

THEOREM II.

Given a function of n variables,

$$F(x^1, x^2, x^3 \dots, x^n),$$

continuous with continuous non-vanishing first derivatives in the region defined, then in order that there shall exist a unique function $\mathcal{E}(F)$ such that

$$\xi(F) \equiv \sum_i u^i(x^i),$$

it is necessary and sufficient that

$$\frac{F_{x^i}}{F_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Proof:

Necessary conditions:—

If the ξ function exists then the functional matrix

$$\left\| \begin{array}{cccccc} u^1_{x^1} & u^2_{x^2} & u^3_{x^3} & \cdots & \cdots & u^n_{x^n} \\ F_{x^1} & F_{x^2} & F_{x^3} & \cdots & \cdots & F_{x^n} \end{array} \right\|$$

must be of rank one. Hence

$$\frac{F_{x^i}}{F_{x^j}} = \frac{u^i_{x^i}}{u^j_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Sufficient conditions:—

We assume that

$$\frac{F_{x^i}}{F_{x^j}} = \alpha^i(x^i)\,\alpha^j(x^j).$$

Then we have the following identities:

a) $$\frac{\partial^2}{\partial_{x^i}\,\partial_{x^j}} \log \frac{F_i}{F_j} = 0,$$

b) $$\frac{F_i}{F_j} = \frac{F_{ik}}{F_{jk}} = \frac{F_{ikl}}{F_{jkl}},$$

for $k, \neq i$ or j and

c) $$F_{ij}\left\{F_{ki}F_i - F_{ii}F_k\right\} = F_i\left\{F_{ijk}F_i - F_{iij}F_k\right\},$$

where for convenience of notation,

$$F_i \equiv F_{x^i}, \quad F_{ik} = F_{x^i x^k}, \qquad \text{etc.}$$

Making use of a), b), and c), it is easily shown that the functional matrix

$$\begin{Vmatrix} \frac{\partial}{\partial x^1}\left(\frac{F_{ij}}{F_i F_j}\right) & \frac{\partial}{\partial x^2}\left(\frac{F_{ij}}{F_i F_j}\right) & \cdots & \frac{\partial}{\partial x^n}\left(\frac{F_{ij}}{F_i F_j}\right) \\ F_1 & F_2 & \cdots & F_n \end{Vmatrix}$$

is of rank one. It follows that

d) $$\frac{F_{ij}}{F_i F_j} = \lambda(F).$$

Now the differential equation that defines the ξ function is

$$\xi_{ij} \equiv \xi_{FF} F_i F_j + \xi_F F_{ij} = 0,$$

or

$$\frac{\xi_{FF}}{\xi_F} = -\frac{F_{ij}}{F_i F_j}, \quad = -\lambda(F) \text{ from d).}$$

Hence $\xi(F)$ is uniquely determined, namely,

$$\xi(F) = K\int e^{-\int\lambda(F)dF} dF + H.$$

where K and H are constants of integration.

Now, for our problem, if F satisfies the given conditions, we can apply the two theorems in succession and we have that there exists a unique function

$$\xi(F) \equiv u^i(\ell_i).$$

But due to the symmetry of F all the u^i functions will be the same and we have

$$\xi(F) \equiv u(\ell_i).$$

If we now define

$$\varphi(p, \ell_i) = \frac{1}{n}\xi(p) - u(\ell_i),$$

we have

$$\sum_i g(p, t_i) = \xi(p) - \xi(F) = 0.$$

4. *General Error Function*

We may now write equations (1) and (2) in the form respectively

$$\sum_i \frac{\partial}{\partial p} \log \varphi(p, t_i) = 0,$$

$$\sum_i g(p, t_i) = 0.$$

These equations must be simultaneously satisfied for an arbitrary set of values t_i in the region defined. It follows that they are identical. Thus

$$\frac{\partial}{\partial p} \log \varphi(p, t_i) = \psi(p) g(p, t_i),$$

where $\psi\ (p)$ is an arbitrary function.
Integrating, we get

$$\varphi(p, t_i) = \theta(t_i) e^{\int \psi(p) g(p, t_i) dp} \tag{3}$$

where $\theta(t_i)$ is an arbitrary function. This is our general error function. In order to insure a maximum we must have

$$\psi(p) g_p \ngtr 0. \tag{4}$$

5. *A Generalised Normal Function*

If we now make the additional assumption that

$$\varphi(p, t_i) \equiv \phi\{g(p, t_i)\},$$

we have

$$\begin{vmatrix} \varphi_p & \varphi_{t_i} \\ g_p & g_{t_i} \end{vmatrix} = 0$$

Expanding and simplifying, we get

$$\frac{\theta_{\ell_i}}{\theta g_{\ell_i}} = \frac{\psi(p)\, g(p, \ell_i)}{g_p} - \int \psi(p)\, dp.$$

Differentiating with respect to ℓ_i, we get

$$\frac{1}{g_{\ell_i}} \frac{\partial}{\partial \ell_i} \left(\frac{\theta_{\ell_i}}{\theta g_{\ell_i}} \right) = \frac{\psi(p)}{g_p} = k.$$

Integrating and substituting in (3), we get

$$\varphi(g) = c e^{kg^2}$$

From (4) we have

$$Kg_p^2 \not> 0,$$

Hence

$$\varphi(g) \doteq c e^{-h^2 g^2}. \tag{5}$$

We shall refer to this function as the "Generalized Normal Error Function".

5. *Application to Special Cases*

If p is defined as the arithmetic mean, then the region considered is $-\infty < \ell_i < +\infty$, and

$$g(p, \ell_i) = p - \ell_i.$$

The normal law is obtained directly from (5), and from (4) we have

$$\varphi(p, \ell_i) = \theta(\ell_i)\, e^{\int \psi(p)(p - \ell_i)\, dp}$$
$$= \theta(\ell_i)\, e^{w(p) + \ell_i v(p)},$$

where

$$\frac{dW}{dp} + p\,\frac{dV}{dp} = 0,$$

which is the same as the Poincaré function.

For the geometric mean, the region considered is

$$0 < l_i < \infty$$

and

$$g(p, l_i) = \log p - \log l_i.$$

Hence, from (3)

$$\Phi(p, l_i) = \theta(l_i)\, e^{\int \psi(p)\{\log p - \log l_i\} dp},$$

and from (4)

$$\Phi\{\log p - \log l_i\} = c e^{-h^2\{\log p - \log l_i\}^2}.$$

The Geometric mean as the most probable value, as well as its generalized normal curve are used for certain astronomical photometric measurements.[1]

For the harmonic mean, the region considered is

$$(0 < l_i < \infty)$$

and

$$g(p, l_i) = \left(\frac{1}{p} - \frac{1}{l_i}\right).$$

Then from (3), we have

$$\Phi(p, l_i) = \theta(l_i)\, e^{\int \psi(p)\{\frac{1}{p} - \frac{1}{l_i}\} dp}$$

and from (4), we have

$$\Phi\left\{\frac{1}{p} - \frac{1}{l_i}\right\} = c e^{-h^2\{\frac{1}{p} - \frac{1}{l_i}\}^2}.$$

7. *Remarks About the Generalised Normal Curves*

Let us consider briefly some characteristics of the generalized normal curves corresponding to the following three special cases.

---

[1] Whittaker & Robinson, Calculus of observations (1924), p. 218.

(a) *Arithmetic mean:* Here

$$\phi(p,l) = ce^{-h^2(p-l)^2}$$

From this equation we see that

$$\phi(p,p+\epsilon^2) = \phi(p,p-\epsilon^2),$$
$$\phi(p,l) = \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = ce^{-h^2p^2},$$
$$\phi(p,\infty) = 0.$$

(b) *Harmonic mean:* In this case

$$\phi(p,l) = ce^{-h^2\{\frac{1}{p}-\frac{1}{l_i}\}^2},$$

from which we see that

$$\phi(p,p-\epsilon^2) < \phi(p,p+\epsilon^2)$$
$$\phi(p,l) < \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = 0,$$
$$\phi(p,\infty) = ce^{-\frac{h^2}{p^2}}$$

(c) *Geometric Mean:* Here

$$\phi(p,l) = ce^{-h^2\{\log p - \log l_i\}^2}$$

and

$$\phi(p,p-\epsilon^2) < \phi(p,p+\epsilon^2),$$
$$\phi(p,l) < \phi(p+\epsilon^2, l+\epsilon^2),$$
$$\phi(p,0) = 0,$$
$$\phi(p,\infty) = 0,$$

Instead of treating these normal functions as three distinct error laws referred to the same measuring scale, we can regard them as a single error law with reference to three different measuring scales (see sketch). This viewpoint helps to explain the above mentioned characteristics of these laws.

The law for the arithmetic mean applies when an object is measured with a uniformly graduated scale in p. The characteristics for this law follow directly from the consideration that the scale is everywhere the same.

The law for the harmonic mean holds when an object is measured with a reciprocally graduated scale, as for instance measuring the volume of a gas with a pressure gauge graduated for volume. In this case the scale becomes crowded as p increases, and hence

$$\Phi(p+\epsilon^2, t+\epsilon^2) > \Phi(p, t),$$

and also

$$\Phi(p, p+\epsilon^2) > \Phi(p, p-\epsilon^2).$$

For large values of p it would take only a small error in the reading of the scale to make an infinitely large error in the value of p and hence $\Phi(p, \infty)$ does not necessarily vanish. On the other hand the zero point is at an infinite distance and hence $\Phi(p, 0) = 0$.

The law for the geometric mean holds for measuring objects with a logarithmically graduated scale. The same remarks as for the harmonic mean apply here, except that in this case it would take an infinitely large error in the reading of the scale to make an infinitely large error in the value of p. Hence $\Phi(p, \infty) = 0$.

8. *The Pearson Curves*

Leaving out the subscripts in (3), we have for the general error function

$$\Phi(p, t) = \Theta(t) e^{\int \psi(p) g(p, t)\, dp}$$

Remembering that

$$g(p,\ell) \equiv \frac{1}{n}\xi(p) - u(\ell),$$

we have

$$\frac{\partial \phi}{\partial \ell} = \phi\left\{\frac{\theta_\ell}{\theta} + u(\ell)\int \psi(p)\,dp\right\}.$$

Thus for a given p the curve approaches the ℓ axis asymptotically

Let us now impose the condition that

$$\left.\frac{\partial \phi}{\partial \ell}\right|_{\ell = p} = 0,$$

then

$$\frac{\theta_\ell}{\theta} = u(\ell)\int \psi(\ell)\,d\ell.$$

Integrating, we get

$$\theta(\ell) = ce^{\int u(\ell)\left\{\int \psi(\ell)\,d\ell\right\}d\ell}$$

so that

$$\phi(p,\ell) = ce^{-\int u(\ell)\left\{\int \psi(\ell)\,d\ell\right\}d\ell + \int \psi(p)g(p,\ell)\,dp} \tag{6}$$

and

$$\frac{\partial \phi}{\partial \ell} = \phi u_\ell \int_\ell^p \psi(t)\,dt.$$

where t is a variable of integration. If we now take as a special case

$$\psi(t) = 1,$$
$$u_\ell = \left[b_0 + b_1\ell + b_2\ell^2 + \qquad\right]^{-1}.$$

we have

$$\frac{\partial \varphi}{\partial t} = \varphi \frac{p-t}{b_0+b_1 t+b_2 t^2},$$

which is the differential equation defining the Pearson system of frequency curves. For this case, (6) reduces to

$$\varphi(p,t) = ce^{\int \frac{p-t}{b_0+b_1 t+b_2 t^2}dt + \frac{1}{n}\int \xi(p)dp}$$

from which we see that by a proper choice of $\xi(p)$ we can choose the value of p for which the product $\prod_i \varphi(p,t_i)$ shall be a maximum.

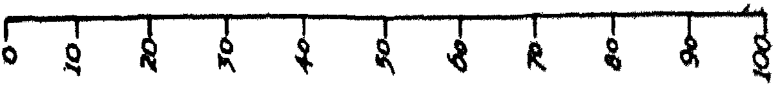
It may be noted that the differential equation defining the Pearson curves is often derived on the basis of the assumptions that the curve shall approach the t axis asymptotically, and have only one maximum point.

In conclusion, it appears that if we restrict the function that defines to satisfy the assumptions given in this paper, and also impose the condition that p shall be the most probable value in the sense, that the product

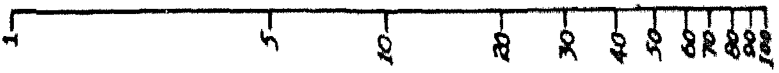
$$\prod_i \varphi(p,t_i)$$

shall be a maximum with respect to p, then (3) is the most general form of the error function.

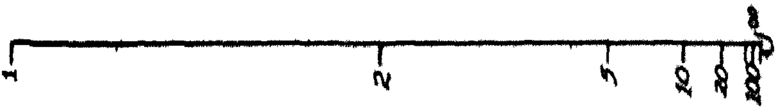
Scales, corresponding to the arithmetic, geometric, and harmonic means.



Arithmetic Mean — Uniform Scale



Geometric Mean — Logarithmic Scale



Harmonic Mean — Reciprocal Scale

albert Wertheimer

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A COEFFICIENT OF LINEAR CORRELATION BASED ON THE METHOD OF LEAST SQUARES AND THE LINE OF BEST FIT.

By J. B. Coleman

Given N points in a plane, corresponding to N pairs of values for two variables, X and Y, we find the line of best fit and the line of *worst* fit, by the method of least squares*. Then we derive a cofficient of correlation based on the sum of the squares of the distances of the points from these two lines.

The line of best fit is in the line such that the sum of the squares of the distances of the points from it is a minimum. The line of *worst* fit is the one from which the sum of the squares of the distances of the points is a maximum. We shall refer to them, respectively, as the *minimum* and *maximum* lines.

For convenience we take the origin at the centre of gravity of the points, letting x and y denote deviations of X and Y, respectively, from their arithmetic means.

1. The two lines pass thru the arithmetic means of the X's, and of the Y's.

$y = mx + b$ may represent any line of the plane. The distance, d_i, of a point, (x_i, y_i), from the line is $\frac{y_i - mx_i - b}{\sqrt{1+m^2}}$. The sum of the squares of the distances of the N points from the line will be

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*For a general discussion of this method of fitting when q variables are involved, see, Pearson, Karl, "On Lines and Planes of Closest Fit to Systems of Points in Space", Phil. Mag., 6th series, vol. ii, 1901, P. 559.

$$(1) \quad \Sigma \underline{d}^2 = \frac{\Sigma y^2 + m^2 \Sigma x^2 + N b^2 - 2m \Sigma xy - 2b \Sigma y + 2mb \Sigma x}{1 + m^2}$$

Using $\Sigma x = \Sigma y = 0$, in the condition for maximum or minimum values in (1), the condition reduces to $\underline{b} = 0$, and the theorem follows.

2. To find the slopes of the two lines.
Equation (1) now becomes

$$(2) \quad \Sigma \underline{d}^2 = \frac{\Sigma y^2 - 2m \Sigma xy + m^2 \Sigma x^2}{1 + m^2}.$$

The condition under which (2) will have a maximum or minimum values, is that

$$m^2 \Sigma xy + m(\Sigma x^2 - \Sigma y^2) - \Sigma xy = 0.$$

This condition is satisfied by two values of $\underline{m}$, namely;

$$(3) \quad m_1 = \frac{\Sigma y^2 - \Sigma x^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2 \Sigma xy},$$

$$(4) \quad m_2 = \frac{\Sigma y^2 - \Sigma x^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2 \Sigma xy}.$$

It is found by considering the second derivative that (4) is the condition under which $\Sigma \underline{d}^2$ will have a maximum value, and (3) is the condition for a minimum value.

The equation of the *minimum* line is $y = m_1 x$, and that of the *maximum* line is $y = m_2 x$. The value of m_1 and m_2 are those given in (3) and (4).

3. The *maximum* and *minimum* lines are perpendicular to each other.

That $m_1 = -1/m_2$ is easily shown from (3) and (4).

Further m_1 has the same sign as Σxy, and m_2, the opposite sign, since their numerators are, respectively, positive and negative.

4. The *minimum* line lies between the two lines of regression, or coincides with them.

If $\Sigma xy > 0$,

$$m_1 \leqq \frac{\Sigma y^2 - \Sigma x^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4\Sigma x^2 \Sigma y^2}}{2\Sigma xy}$$

since $\Sigma x^2 \Sigma y^2 \geqq (\Sigma xy)^2$.

Hence $m_1 \leqq \Sigma y^2 / \Sigma xy = m_{xy}$, the slope of the line of regression of x on y.

Rationalizing the numerator of (3), and noting that $\Sigma x^2 \Sigma y^2 \geqq (\Sigma xy)^2$, we obtain

$$m_1 \geqq \frac{2\Sigma xy}{\Sigma x^2 - \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4\Sigma x^2 \Sigma y^2}} = \frac{\Sigma xy}{\Sigma x^2} = m_{yx},$$

the slope of the line of regression of y on x.

In the same way it may be shown that if $\Sigma xy < 0$, then $m_{xy} \leqq m_1 \leqq m_{yx}$.

The condition that m_1 be equal to the slope of one line of regression is the same that it be equal to the other, so that the *minimum* line coincides with both lines of regression, or else lies between the two.

5. To find the sum of the squares of the distances of the points from the *minimum* line; also from the *maximum* line.

Let d be the distance of a point from the line $y = m_1 x$

$$\Sigma d^2 = \Sigma\left(\frac{y - m_1 x}{\sqrt{1 + m^2}}\right)^2 = \frac{\Sigma y^2 - 2 m_1 \Sigma xy + m_1^2 \Sigma x^2}{1 + m_1^2}.$$

Substituting for m from (3) and reducing, we obtain

$$(5) \qquad \Sigma d^2 = \left[\Sigma x^2 + \Sigma y^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}\right]/2.$$

Similarly, if D represents the distance of a point from the line $y = m_2 x$, we obtain

$$(6) \qquad \Sigma D^2 = \left[\Sigma x^2 + \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}\right]/2.$$

6. To find a coefficient of linear correlation.

Let $q = \sqrt{\Sigma d^2 / \Sigma D^2}$ Substituting from (5) and (6), and reducing, we obtain

$$(7a) \qquad q = \frac{\Sigma x^2 + \Sigma y^2 - \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}{2\sqrt{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2}}, \text{ or}$$

$$(7b) \qquad q = \frac{2\sqrt{\Sigma x^2 \Sigma y^2 - (\Sigma xy)^2}}{\Sigma x^2 + \Sigma y^2 + \sqrt{(\Sigma x^2 - \Sigma y^2)^2 + 4(\Sigma xy)^2}}$$

q represents the ratio of the root-mean-squares of the distances of the point from the *minimum* and *maximum* lines. This ratio is a measure of the closeness of fit of the points to a line, and should furnish a measure of linear correlation. The value of q may vary from 0 to 1. $q = 0$ indicates that the points are all on a straight line, hence that the correlation is perfect. It is of interest to note that when $q = 0$, (7b) gives $\Sigma xy/N\sigma_x\sigma_y$ $[= r] = \pm 1$. When q is 1 the mean squares of the distances of the points from two lines at right angles is the same, and linear correlation is lacking. Hence $1 - q$ would furnish a coefficient conforming to the customs that it have the value 1 for perfect correlation, and 0 for lack of correlation.

Values of q found from (7a) or (7b) would involve the units in which X and Y are given. Hence these forms would be objectionable, in that q could be made to assume different values for the same data, by changing the units in which x and y are expressed. However, this objection may be removed by taking σ_x and σ_y as units in which to express X and Y. (7b) then reduces to

$$q = \frac{\sqrt{N^2 - (\Sigma xy)^2/\sigma_x^2\sigma_y^2}}{N + |\Sigma xy/\sigma_x\sigma_y|}.$$

The coefficient $1 - q$ may now be expressed as

$$r_c = 1 - q = 1 - \sqrt{\frac{N - |\Sigma xy/\sigma_x\sigma_y|}{N + |\Sigma xy/\sigma_x\sigma_y|}}. \tag{8}$$

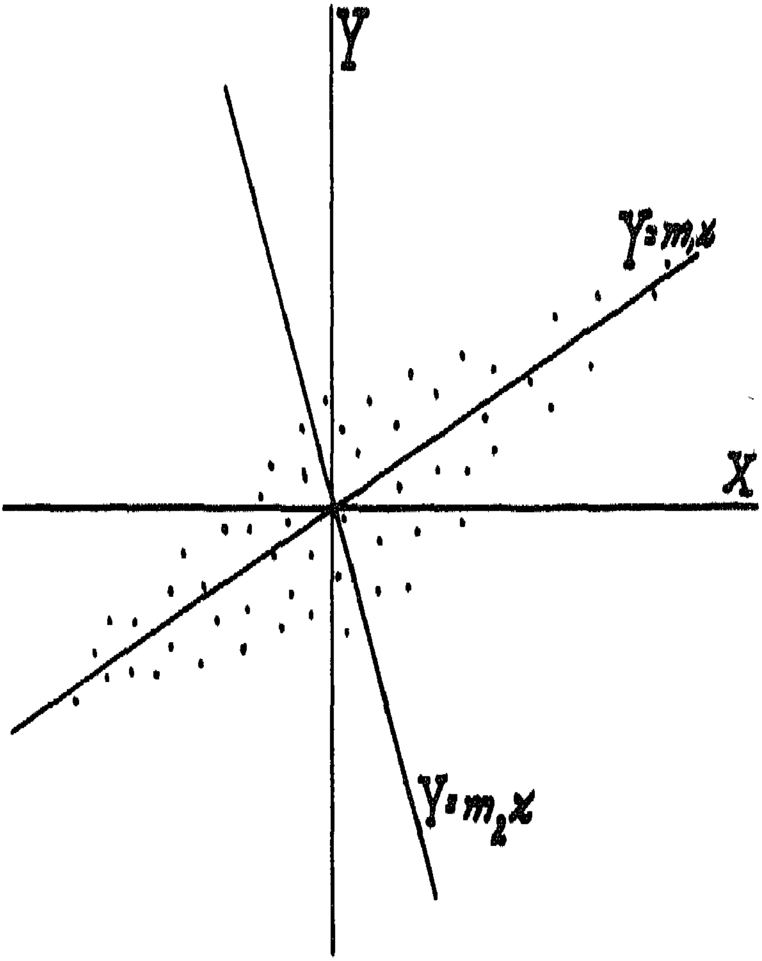
The sign of the coefficient should agree with that of the slope of the line to which the points are fitted. Hence, when the value for $1 - q$ has been found it should be given the same sign as the slope of the *minimum* line. But the slope of the *minimum* line is determined by that of Σxy, so that the sign given to $1 - q$ should be that of Σxy.

The coefficient $1-q$ may be expressed immediately in terms of the Pearson coefficient, r, which is equal to $\Sigma xy/N\sigma_x\sigma_y$. Making this substitution in (8) we have

$$r_c = 1-q = 1-\sqrt{\frac{1-|r|}{1+|r|}}\ .$$

In the table are shown values of $1-q$ corresponding to some given values of r. The values for $1-q^2$ have also been listed corresponding to the same set of values for r. The maximum difference occurs between $1-q$ and r when $r=.839$ and $1-q=.704$, a difference of .135 by which $1-q$ is smaller.

r	$r_c = 1-q$	$1-q^2$
1	1	1
.99	.929	.995
.95	.840	.974
.9	.771	.947
.839	.704	.912
.8	.667	.889
.7	.580	.824
.6	.500	.750
.5	.423	.667
.4	.345	.571
.3	.266	.462
.2	.183	.333
.1	.095	.182
0	0	0



Y
X
$Y = m_1 x$
$Y = m_2 x$

A STUDY OF THE DISTRIBUTION OF MEANS ESTIMATED FROM SMALL SAMPLES BY THE METHOD OF MAXIMUM LIKELIHOOD FOR PEARSON'S TYPE II CURVE

By John L. Carlson

The object of this paper is to study the distribution of estimates of the parameter of location for Pearson's Type II Curve, estimated by the method of maximum likelihood from small samples.

R. A. Fisher has assumed,[1] and Professor Hotelling has proved[2] that in large categories of cases the distribution of an optimum statistic approaches normality as the sample size increases. This normality has been assumed to hold for optimum statistics in general whether calculated from large samples or small ones, and it has also been assumed that optimum statistics have minimum variance and always give better fits than do statistics calculated by the method of moments. That this is the case whenever the sample is large and the distribution of optimum statistics normal is made plausible by the reasoning of R. A. Fisher.[3] In case the sample is small, however, there may be reason to doubt that the normality of distribution of optimum statistics holds, and that the other conclusions hold. It is with this phase of the subject that we shall be concerned in what follows.

Before entering into the topic under discussion it will be convenient to review some of the more elementary facts regarding the curve with which we are to be concerned. We shall take first the general equation for the curve in the form,

$$(1) \qquad y = y_0\left(1 - \frac{(x-m)^2}{a^2}\right)^p$$

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[1] On the Mathematical Foundations of Theoretical Statistics, R. A. Fisher, Phil. Trans. Series A, Vol. 222, 1922. Pp. 309-368.

[2] The Consistency and Ultimate Distribution of Optimum Statistics. Harold Hotelling, Trans. Amer. Math. Soc. Vol. 32, No. 4. Pp. 847-859.

[3] Ibid, Pp. 328-368.

and determine the effect of variation of the constants.

When $p=0$ the equation reduces to the straight line $y=y_0$

When $p=+1$ the equation is that of a parabola with $y=y_0$ at the point $x=m$ and the x intercepts at the points $x=\pm a$. It of course meets the x axis at an angle.

When $p=+2$ the equation $y=0$ is of the fourth degree in x with double roots at the points $x=\pm a$.

In general when $p=n$ the equation $y=0$ is of degree $2n$ in x and has two sets of n-fold multiple roots $x=\pm a$.

Since our curve is to be a probability curve it will of necessity have unit area, and this fact makes it possible for us to evaluate y_0 in terms of the parameters a and p. In order to do this we shall perform the integration below.

$$\text{Area} = 1 = \int_{x-a}^{x+a} y\,dx = 2y_0\int_0^a \left(1-\frac{x^2}{a^2}\right)^p dx$$

whence

$$1 = 2ay_0\int_0^{\frac{\pi}{2}} \cos^{2p+1}\theta\,d\theta$$

but now since

$$(2) \qquad \int_0^{\frac{\pi}{2}} \cos^n\theta\,d\theta = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

we have $\quad 1 = 2ay_0\dfrac{\sqrt{\pi}}{2}\,\dfrac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}$

and so

$$y_0 = \frac{\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)}$$

Therefore

$$(3) \qquad y = \frac{\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)}\left[1-\frac{(x-m)^2}{a^2}\right]^p$$

Formula (2) we shall find to be of value a number of times and formula (3) is the form in which equation (1) will be used throughout the remainder of the paper.

It will be worth while now to consider the likelihood function L together with its first and second partial derivatives with respect to m. (We shall hereafter refer to m as the parameter of location, a as the parameter of scaling, and p as the parameter of shape.) We are to use m to denote the estimate of m obtained by the method of maximum likelihood, in accordance with the convention introduced by Fisher,[1] and it will be with this parameter that we shall concern ourselves in this investigation.

We have from (3) on the preceding page

$$(4) \qquad L = n \log \frac{\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} + p\sum_{i=1}^{n} \log\left[1-\frac{(x_i-m)^2}{a^2}\right]$$

and so

$$(5) \qquad \frac{\partial L}{\partial m} = 2p\sum_{i=1}^{n} \frac{x_i-m}{a^2-(x_i-m)^2}$$

and

$$(6) \qquad \frac{\partial^2 L}{\partial m^2} = -2p\sum_{i=1}^{n} \frac{a^2+(x_i-m)^2}{[a^2-(x_i-m)^2]^2}$$

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[1] Ibid Pp. 309-368.

At this point we shall stop to consider the effect of variation of the parameters a and p upon our estimate of $\hat{m}$. Let us first consider p . Since the method of maximum likelihood is here merely the method of the differential calculus it follows from a consideration of equation (5) that our estimate of $\hat{m}$ will be independent of p . for any particular sample. Such is not the case when we consider a , however, for any change in a allows a change in the variance of $\hat{m}$ for the particular sample.

We shall find it advantageous to cover as much of the theoretical work as possible before embarking upon our experimental check and its great amount of numeral calculations, for it is only by means of a check between theory and experiment that we are able in the present state of knowledge, to judge of the applicability of the method of maximum likelihood to small samples. Of course it will be necessary to consider the distribution of our estimates of $\hat{m}$, in order to make this statistic of practical use, and it is desirable to know the theoretical variances of $\bar{x}$ the arithmetic mean, $\hat{m}$, and the experimentally obtained variance of the distribution of our estimates of $\hat{m}$. The first of these we can obtain from theory, the second from an approximation valid only in the limit, and the last by means of calculation based on actual sampling.

We shall be concerned first with the theoretical variance of $\bar{x}$. It is well known that the variance of $\bar{x}$ is equal to the variance of the distribution divided by n . We must first, therefore, find the variance of the distribution.

From (3) page 88 it follows that

$$\sigma^2 = \frac{2\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^p dx$$

$$\sigma^2 = \frac{2\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)}\int_0^{\frac{\pi}{2}}\left[\cos^{2p+1}\theta-\cos^{2p+3}\theta\right]d\theta$$

remembering now (2) page 87

$$\sigma^2 = \frac{a^2}{2p+3}$$

whence it follows that

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{a^2}{n(2p+3)} . \tag{7}$$

We shall now calculate the limiting form of the variance of $\hat{m}$. Fisher has proved[1] that if the distribution of optimum statistics is normal the variance of an optimum statistic is equal to the negative reciprocal of the mathematical expectation of the second partial derivative of the logarithm of the likelihood with respect to the parameter in question.

We may write, therefore

$$-\frac{1}{\sigma_{\hat{m}}^2} = \frac{-4np\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} \int_0^a \frac{(a^2+x^2)}{(a^2-x^2)} \left[1-\frac{x^2}{a^2}\right]^p dx$$

whence

$$-\frac{1}{\sigma_{\hat{m}}^2} = \frac{-4np\Gamma(p+\frac{3}{2})}{a\sqrt{\pi}\,\Gamma(p+1)} \int_0^{\frac{\pi}{2}} (2\cos^{2p-3}\theta - \cos^{2p-1}\theta)\,d\theta$$

and so again referring to (2) page 87 we have

$$-\frac{1}{\sigma_{\hat{m}}^2} = -\frac{2np(p+\frac{1}{2})}{a^2(p-1)}$$

hence we have

$$\sigma_{\hat{m}}^2 = \frac{a^2(p-1)}{np(2p+1)} \tag{8}$$

The efficiency of the mean is then

$$E = \frac{(p-1)(2p+3)}{p(2p+1)} = \frac{\sigma_{\hat{m}}^2}{\sigma_{\bar{x}}^2} \tag{9}$$

The next problem with which we must concern ourselves is that of experimental verification of the assumptions under discussion. The problem is briefly that of choosing a number of small samples from a population which obeys our law of frequency, estimating m for these samples, and then calculating the variance for the distribution. Also we shall draw an histogram of the distribution and observe the general type of the distribution, so nearly as that is possible from our samples.

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[1] Ibid Pp. 327-328

The problem of choosing our samples is not the least of our difficulties, for we can not take all types of samples. They must be of a very special nature: they must be from a population of the Type II. In order to accomplish this it will be necessary to have a table of areas corresponding to given values of x for the Type II Curve. There is no such table available to the knowledge of the writer, and it is therefore necessary to construct the table before we can procede with the choosing of the samples. After the table has been built, we can with the aid of Tippett's Tables,[1] choose our samples with ease. The manner of choosing is as follows. Take the numbers from Tippett's Tables as areas under the Type II Curve and look up in the table of areas the values of x corresponding to the smallest area containing the area found from the Random Numbers. This will give the value of x to be taken. Since we will take four digits let the fifth digit determine the sign. If it is odd take the sign $-$, if it is even take the sign $+$.

There are two ways in which a table of this nature can be prepared, and the method employed must in any case be determined by the degree of accuracy attainable and the amount of labor involved. One of these methods is that of the calculus of finite differences, determining the zero order differences by means of algebra, and from these by the process of addition building up the table. This method is best used when a dependable listing adding machine is at hand. The other method is that of direct integration, and it is found that with the aid of two calculating machines this is by far the quicker. It was this method that was applied in the building of the table on page 92 and 93.

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[1] **Tracts For Computers, No. XV.**

Table of Areas Under Pearson's Type II Curve, Correct to 9 Places of Decimals. The Areas are included between ordinates located $\pm x$ units from the parameter of location.

Constants. $y_0 = {}^{15}/_{16}\,a$, $m = 0$, $p = 2$, and $a = 1$

x	Area	x	Area
0.00	0.000 000 000	30	529 661 250
01	018 748 750	31	545 084 843
02	037 490 001	32	560 298 291
03	056 212 259	33	575 295 077
04	074 920 038	34	590 073 828
05	093 593 867	35	604 625 820
06	112 230 292	36	618 947 482
07	130 821 880	37	633 034 148
08	149 361 229	38	646 881 319
09	167 840 964	39	660 484 657
10	186 253 750	40	673 840 000
11	204 592 289	41	686 943 358
12	222 849 331	42	699 790 921
13	241 017 673	43	712 379 067
14	259 090 168	44	724 704 358
15	277 059 727	45	736 763 555
16	294 919 322	46	748 553 612
17	312 661 995	47	760 071 688
18	330 280 859	48	773 151 488
19	347 769 104	49	782 281 572
20	365 120 040	50	792 968 750
21	382 326 904	51	803 374 696
22	399 383 261	52	813 497 651
23	416 282 613	53	823 336 081
24	433.018 598	54	832 888 688
25	449 584 961	55	842 154 414
26	465 975 552	56	851 132 442
27	482 184 334	57	860 009 702
28	498 205 389	58	868 223 379
29	514 032 918	59	876 335 911
30	529 661 250	60	884 160 000

61	892 546 290	81	985 203 165
62	898 944 981	82	987 317 441
63	905 907 620	83	989 230 274
64	912 585 318	84	990 949 478
65	919 120 273	85	992 483 242
66	925 092 472	86	993 840 132
67	930 925 941	87	995 029 095
68	936 482 509	88	996 059 469
69	941 764 926	89	996 940 979
70	946 776 250	90	997 683 750
71	951 519 851	91	998 298 304
72	955 999 411	92	998 795 571
73	960 294 310	93	999 186 888
74	964 182 748	94	999 484 008
75	967 895 508	95	999 699 102
76	971 362 202	96	999 844 762
77	974 588 156	97	999 934 010
78	977 579 039	98	999 980 299
79	980 340 865	99	999 997 519
80	982 880 000	100	1.000 000 000

The table on the preceding page was built by direct integration of (3) page 88, with $p=2$ and $a=1$ These values for the parameters were chosen so as to save as much labor in calculation as possible, and at the same time maintain the desired shape of the curve. There has of course been no less of generality in setting $a=1$ but we have limited ourselves quite definitely in using the value $p=2$. The accuracy of the table to 9 places of decimals has been assured by calculating all values to 13 places and then determining the maximum error over the whole range which was 625 in the 13 place due to the use of the decimal equivalent of $^2/_3$. in the third degree term.

Our table of areas being complete and the problem of sampling thus solved, we must next consider the task of estimating $\hat{m}$. Since we have already taken $a=1$ and $p=2$ in building our tables we shall continue to use these values throughout the work.

The next question to be settled before beginning our work is the size of the samples with which we are to deal. The case $m = 1$ is of course of no interest for the best estimate of a single observation is the observation itself. The case $m = 2$ is likewise of no interest for in this case the arithmetic mean coincides with the solution by the method of maximum likelihood. Therefore it is the case $m = 3$ with which we shall be concerned. Our results are not then trivial, and at the same time we have the case which is the easiest to deal with, since the number of numerical calculations for each sample is reduced to the minimum.

Before going ahead with any attempt at solving an equation suchas (5) page88., it is well to have in mind a picture of what we are actually trying to accomplish. With such a picture in mind we are better able to realize the difficulties of the situation and so are better able to cope with them. To this end we have included a graph of the problem involved which will make clear at a glance just what must be done to find the true value of m.

We have drawn, page 96 , from plotted points the curves representing L , $\frac{\partial L}{\partial m}$ and the three terms of $\frac{\partial L}{\partial m}$. Also we have drawn in the asymptotes to the curve which are of significance. If we are able to find the point at which $\frac{\partial L}{\partial m} = 0$ we have the solution to our problem. This involves solving a fifth degree equation. We shall use Newton's method of successive approximation. Fisher states[1] that in some cases at least, we may start with an inefficient statistic and by a single approximation obtain an efficient one. Whether or not this is the case for small samples we shall see when our calculations have been analyzed.

It will be seen upon examining the graph that as is allowed to vary each of the therms of $\frac{\partial L}{\partial m}$ varies from $-\infty$ to $+\infty$, and so their sum also varies between these limits. It will be seen that the asymptote corresponding to the largest allowable value of m can be found by adding the value of a to the smallest observation, and that the asymptote corresponding to the smallest allowable value of m may be found by subtracting the value of a from the greatest observation. It is thus evident that as dispersion in the sample increases the variance of m must surely decrease. That this fact is of fundamental importance in choosing our first estimate of m will be seen from consideration of the following case, since it is well known that in the case of a curve with a real finite pole Newton's method may lead us to erroneous results. Let us consider the sample consisting of the three observations $x_1 = -.99$, $x_2 = +.99$, $x_3 = +.99$, for which the arithmetic mean is $\bar{x} = +.33$. If now we take as our first approximation $m_1 = \bar{x}$ we will immediately lead ourselves to an erroneous value of m_2, our estimate of $\hat{m}$. That this is the case will be easily seen by means of the check given on page 94 for using $a = 1$ we locate the asymptotes at $x = \pm .01$. The true value of m located between these asymptotes, is not therefore even to be approached should we take $m_1 = \bar{x}$. This is an extreme case and fortunately not to be anticipated very often, or the arithmetic mean would be deprived of any value whatsoever. We should always be sure that the difference between any observation and the value of m that we are using is not greater than the value of a when dealing with the Type II Curve. This holds no matter what our manner of attack may be.

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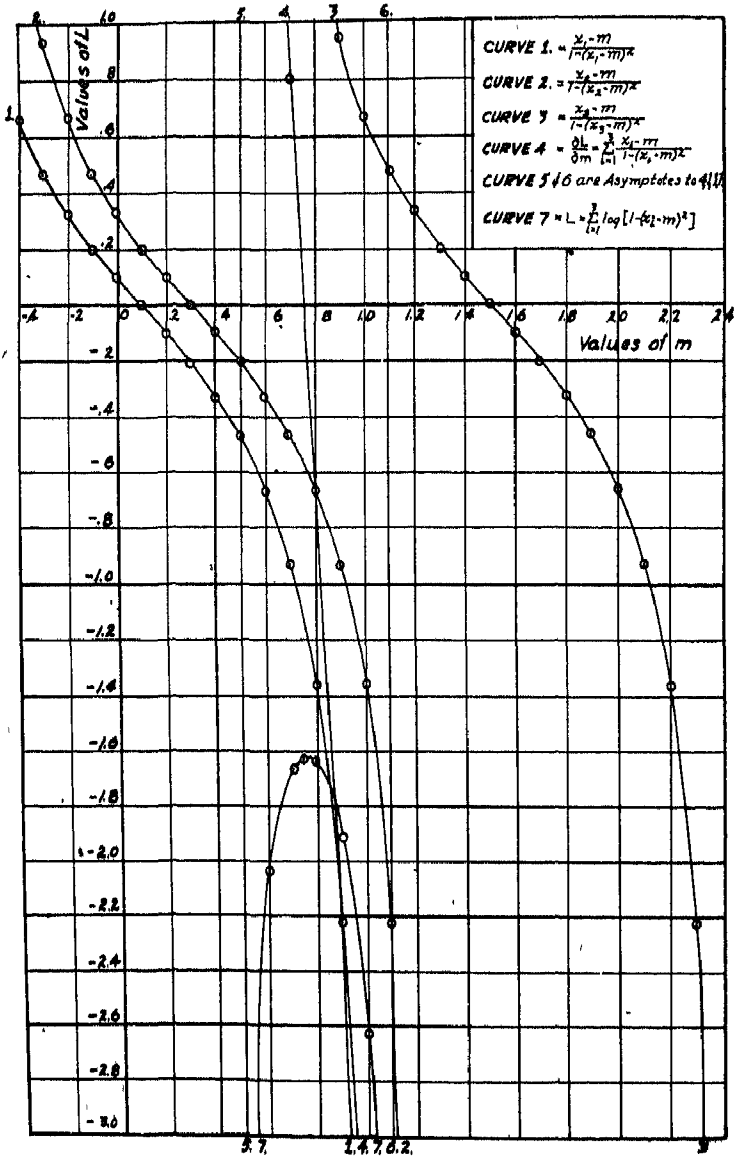
[1] Theory of Statistical Estimation, R. A. Fisher, Proc. Cam. Phil. Soc. Vol. XXII part 5. Pp. 708-709.

We shall now be concerned with the calculations for our 100 samples of 3. All of the data are tabulated in such a way as to be self-explanatory, and so we shall not bother to give sample calculations. The next ten pages cover these calculations. The discussion is continued on page 104.


CURVE 1. = $\frac{x_1 - m}{1-(x_1-m)^2}$
CURVE 2. = $\frac{x_2 - m}{1-(x_2-m)^2}$
CURVE 3 = $\frac{x_3 - m}{1-(x_3-m)^2}$
CURVE 4 = $\frac{\partial L}{\partial m} = \sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$
CURVE 5 & 6 are Asymptotes to 4 & 7
CURVE 7 = $L = \sum_{i=1}^{3} \log[1-(x_i-m)^2]$
Values of L
Values of m
1.0
.8
.6
.4
.2
-.2
-.4
-.6
-.8
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2
-2.4
-2.6
-2.8
-3.0
-.4
-.2
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
1
2
5
4
3
6
5. 7.
1. 4. 7. 6. 2.
3

Sample No	i	x_i	$m_1 = \bar{x}$		$\frac{\partial L}{\partial m}$		$\frac{\partial^2 L}{\partial m^2}$	m_2
			$\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1-(x_i-m)^2}$	$\sum_{i=1}^{3} \frac{x_i-m}{1-(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\sum_{i=1}^{3} -\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
1	1	+.17	-.0700	+ .254667	+.009840	-1.120600	-3.184900	-.066879
	2	-.23		- .164200		-1.051500		
	3	-.15		- .080520		-1.012800		
2	1	-.03	+.0533	- .083882	+.016730	-1.021059	-3.438830	+.049737
	2	-.15		- .212064		-1.133055		
	3	+.34		+ .312676		-1.284716		
3	1	+.67	+.4300	+ .254668	-.177943	-1.190833	-5.080876	+.465022
	2	-.11		- .762281		-2.573777		
	3	+.73		+ .329670		-1.316266		
4	1	-.54	-.2700	- .291230	+.018882	-1.248262	-3.655622	-.264835
	2	-.33		-.060216		-1.010865		
	3	+.06		+.370328		-1.396495		
5	1	-.41	+.0067	-.481688	-.029803	-1.718538	-4.101439	+.013936
	2	+.11		+.104445		-1.032609		
	3	+.32		+.347440		-1.350292		
6	1	-.26	+.0967	-.408701	-.024353	-1.479856	-3.858903	+.103011
	2	+.12		+.023312		-1.001630		
	3	+.42		+.361036		-1.377417		
7	1	+.54	-.2167	+1.770451	+.853498	-4.424677	-9.978709	-.131268
	2	-.50		- .308021		-4.277016		
	3	-.69		- .609932		-1.277016		
8	1	+.19	-.3700	+.815850	+.207992	-2.218101	-4.758692	-.326292
	2	-.64		-.291230		-1.248262		
	3	-.66		-.316628		-1.292329		
9	1	-.12	-.4433	+ .361036	+.011911	-1.377433	-3.666198	-.440051
	2	-.48		- .036749		-1.004049		
	3	-.73		- .312376		-1.284716		
10	1	-.49	-.6533	+ .167774	+.002661	-1.083693	-3.132061	-.652450
	2	-.77		- .118311		-1.041802		
	3	-.70		- .046802		-1.006566		
11	1	-.07	-.1367	+ .066896	-.009546	-1.013446	-3.308885	-.139555
	2	+.04		+.182285		-1.098764		
	3	-.38		-.258727		-1.196675		
12	1	-.40	-.5267	+ .128735	-.005134	-1.049476	-3.182127	-.528283
	2	-.72		- .200836		-1.119217		
	3	-.46		+ .066967		-1.013434		
13	1	-.30	-.2633	- .036749	-.387058	-1.004049	-4.343556	-.352411
	2	-.65		- .454723		-1.589322		
	3	+.16		+ .104414		-1.750185		
14	1	-.14	-.2667	+ .128767	-.010034	-1.049476	-3.263601	-.269775
	2	-.50		- .246729		-1.179312		
	3	-.16		+ .107928		-1.034813		

Sample No	i	x_i	$m_1=\bar{x}$		$\frac{\partial L}{\partial m}$		$\frac{\partial^2 L}{\partial m^2}$	m_2
			$\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1-(x_i-m)^2}$	$\sum_{i=1}^{3}\frac{x_i-m}{1-(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\sum_{i=1}^{3}-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\frac{\partial^2 L}{\partial m^2}m_1-\frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
15	1	-.09	-.2367	+ .149926	+ .002713	- 1.066950	- 3.177717	-.235846
	2	-.20		+ .036749		- 1.004049		
	3	-.43		- .183962		- 1.106718		
16	1	-.29	+.0333	- .360623	- .021856	- 1.376579	- 3.585193	+.039096
	2	+.12		+ .087663		- 1.022996		
	3	+.27		+ .251104		- 1.185618		
17	1	-.10	-.2400	+ .142798	- .000653	- 1.060774	- 3.131187	-.240209
	2	-.39		- .153452		- 1.070113		
	3	-.23		+ .010001		- 1.000300		
18	1	+.26	-.1100	+ .428687	+ .025513	- 1.526159	- 3.878581	-.103422
	2	-.42		- .342958		- 1.341557		
	3	-.17		- .060216		- 1.010865		
19	1	+.23	-.1367	+ .423670	+ .029076	- 1.514353	- 3.830699	-.129111
	2	-.21		- .073695		- 1.016264		
	3	-.43		- .320905		- 1.300082		
20	1	+.37	+.3167	+ .053552	- .025787	- 1.008563	- 4.150227	+.322813
	2	-.09		- .483174		- 1.673112		
	3	+.67		+ .403835		- 1.468552		
21	1	+.21	+.0367	+ .178665	- .983165	- 1.094805	-17.060464	- .094328
	2	-.76		-2.181131		-12.252377		
	3	+.66		+1.019301		- 3.713282		
22	1	-.20	-.0633	- .139303	+ .020958	- 1.057853	- 3.434721	- .057198
	2	+.23		+ .320905		- 1.300082		
	3	-.22		- .160644		- 1.076786		
23	1	-.44	-.3000	- .142798	+ .084156	- 1.060744	- 4.378292	-.280779
	2	-.62		- .356506		- 1.368375		
	3	+.16		+ .583460		- 1.949243		
24	1	-.28	+.1233	- .481639	- .038750	- 1.658197	- 4.063031	- .132837
	2	+.20		+ .077153		- 1.017823		
	3	+.45		+ .365736		- 1.387011		
25	1	-.13	-.2367	+ .107928	+ .155433	- 1.034813	- 3.564903	-.193096
	2	-.56		- .361036		- 1.377417		
	3	-.02		+ .408551		- 1.152673		
26	1	+.10	-.0100	+ .111347	- .014051	- 1.024632	- 3.322761	-.014229
	2	+.14		+ .153452		- 1.070113		
	3	-.27		- .278850		- 1.228016		
27	1	-.79	-.3200	- .603260	- .105746	- 2.011377	- 4.344870	-.344338
	2	-.09		+ .242846		- 1.142660		
	3	-.08		+ .254668		- 1.190833		
28	1	-.08	+.1760	- .273953	+ .001564	- 1.028078	- 3.244534	-.175518
	2	+.43		+ .271517		- 1.216408		
	3	+.18		+ .004000		- 1.000048		

Sample No.	i	x_i	$m_1 = \bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3} -\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	m_2 $\frac{\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
29	1	+.50	-.0767	+ .864077	+.103600	- 2.991572	- 6.332807	-.060341
	2	-.59		- .696923		- 2.329134		
	3	-.14		- .063554		- 1.012101		
30	1	+.02	+.2160	- .203830	+.009519	- 1.123044	- 3.327343	+.213139
	2	+.17		- .046097		- 1.006379		
	3	+.46		+ .259446		- 1.197929		
31	1	-.35	-.2930	- .057185	+.000091	- 1.009800	- 3.058737	-.292970
	2	-.18		+ .114461		- 1.039137		
	3	-.35		- .057185		- 1.009800		
32	1	+.29	-.0100	+ .329670	+.022766	- 1.316266	- 3.456492	-.003414
	2	-.16		- .153452		- 1.070113		
	3	-.16		- .153452		- 1.070113		
33	1	-.45	-.4667	+ .017004	-.000222	- 1.000867	- 3.097560	-.463667
	2	-.60		- .135395		- 1.054671		
	3	-.35		+ .118623		- 1.042022		
34	1	-.23	-.0733	- .160644	+.018610	- 1.076786	- 3.184115	-.067455
	2	+.21		+ .308021		- 1.057853		
	3	-.20		- .128767		- 1.049476		
35	1	-.43	-.0500	- .444132	+.000000	- 1.563278	- 2.832959	-.050000
	2	-.05		+ .000000		- 0.000000		
	3	+.33		+ .444132		- 1.563278		
36	1	-.62	-.1367	- .630593	+6.439597	- 2.100061	-128.928063	-.086753
	2	-.59		- .570533		- 1.909640		
	3	+.80		+7.640723		-124.918362		
37	1	-.44	-.0200	- .509956	+.101753	- 1.734292	- 5.151350	-.000247
	2	+.50		+ .712719		- 2.386551		
	3	-.12		- .101010		- 1.030507		
38	1	-.14	+.1467	- .312376	+.018548	- 1.284716	- 3.730911	+.130837
	2	+.49		+ .387806		- 1.436499		
	3	+.09		- .056882		- 1.009696		
39	1	-.32	+.1900	- .689282	+.059184	- 2.301754	- 5.983818	-.179792
	2	+.74		+ .788530		- 2.677252		
	3	+.15		- .040064		- 1.004812		
40	1	-.17	-.1833	+ .013002	-.001407	- 1.000507	- 3.497606	-.183402
	2	+.08		+ .285611		- 1.233970		
	3	-.46		- .300020		- 1.263129		
41	1	-.30	-.0033	- .325339	-.016626	- 1.308220	- 3.487431	-.008067
	2	+.07		+ .073695		- 1.016264		
	3	+.22		+ .235018		- 1.162947		
42	1	+.01	-.1533	+ .167774	-.006152	- 1.909102	- 4.070338	-.154811
	2	-.10		+ .053451		- 1.008563		
	3	-.37		- .227577		- 1.153673		

Sample No.	i	x_i	$m_1 = \bar{x}$	$\frac{\partial L}{\partial m}$		$\frac{\partial^2 L}{\partial m^2}$		m_2
			$\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\sum_{i=1}^{3} \frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
43	1	+.27	+.0067	+.283036	+.010962	-1.234771	-3.549514	+.003512
	2	+.03		+.023412		-1.001644		
	3	-.28		-.295486		-1.313099		
44	1	-.28	-.2100	-.070344	+.000170	-1.014820	-3.034526	-.209944
	2	-.13		+.080515		-1.019406		
	3	-.22		-.010001		-1.000300		
45	1	-.09	+.0533	-.146304	+.000301	-1.063775	-3.121762	+.053396
	2	+.19		+.139303		-1.057853		
	3	+.06		+.006700		-1.000134		
46	1	-.08	-.1833	+.104414	+.000102	-1.032590	-3.061219	-.183267
	2	-.28		-.097612		-1.028495		
	3	-.19		-.006700		-1.000134		
47	1	-.10	-.3033	+.212064	-.057922	-1.133055	-3.882558	-.318219
	2	-.70		-.470788		-1.630045		
	3	-.11		+.200802		-1.119458		
48	1	+.67	+.1367	+.745508	+.082154	-2.509221	-5.512458	+.121697
	2	+.07		-.066896		-1.013405		
	3	-.33		-.596458		-1.989832		
49	1	-.28	-.2033	-.077153	-.000381	-1.017823	-3.028016	-.203426
	2	-.15		+.053466		-1.008563		
	3	-.18		+.023312		-1.001630		
50	1	-.33	-.0367	-.320905	+.116066	-1.300082	-4.543455	-.011154
	2	+.45		+.637773		-2.123915		
	3	-.23		-.200802		-1.119458		
51	1	+.44	-.1433	+.883184	-.166413	-3.074927	-9.291723	-.160910
	2	-.79		-1.112848		-4.196875		
	3	-.08		+.063251		-2.019971		
52	1	-.15	-.1400	-.010001	+.011845	-1.000300	-5.270074	-.137752
	2	+.36		+.666667		-2.222222		
	3	-.63		-.644821		-2.147552		
53	1	+.09	+.2900	-.208333	-.014143	-1.128472	-3.232369	+.290044
	2	+.31		+.020008		-1.001200		
	3	+.47		+.186027		-1.102697		
54	1	+.21	+.0967	+.114773	-.044559	-1.039349	-3.883929	+.108173
	2	-.29		-.454693		-1.589382		
	3	+.37		+.295361		-1.255198		
55	1	+.39	+.2100	+.186027	-.069261	-1.102697	-4.894084	+.224152
	2	-.21		-.509956		-2.600554		
	3	+.45		+.254668		-1.190833		
56	1	-.30	-.3967	+.097612	-.015355	-1.028495	-3.382457	-.401240
	2	-.22		+.182394		-1.098764		
	3	-.67		-.295361		-1.255198		

			$m_1=\bar{x}$		$\frac{\partial L}{\partial m}$		$\frac{\partial^2 L}{\partial m^2}$	m_2
Sample No.	i	x_i	$\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i-m}{1-(x_i-m)^2}$	$\sum_{i=1}^{3}\frac{x_i-m}{1-(x_i-m)^2}$	$-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\sum_{i=1}^{3}-\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\frac{\partial^2 L}{\partial m^2}m_1-\frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
57	1	-.41	-.1133	-.325339	-.810074	-1.324459	-48.783525	-.129916
	2	-.71		-.926626		-3.371716		
	3	+.78		+4.418971		-44.057350		
58	1	+.04	-.2033	+.258608	-.120557	-1.196675	-4.517265	-.229988
	2	-.69		-.637773		-2.123915		
	3	+.04		+.258608		-1.196675		
59	1	+.43	-.0200	+.564263	-.009196	-1.890704	-4.840247	-.021900
	2	-.48		-.583460		-1.949243		
	3	-.01		+.010001		-1.000300		
60	1	+.69	+.1433	+.779753	+.185625	-2.643417	-5.172942	+.107416
	2	-.17		-.347399		-1.350213		
	3	-.09		-.246729		-1.179312		
61	1	+.12	-.0400	+.164203	-.020191	-1.080198	-3.316004	-.046089
	2	+.09		+.132234		-1.052162		
	3	-.33		-.316628		-1.183644		
62	1	+.63	+.3467	+.308021	-.004016	-1.277016	-3.585766	+.347820
	2	+.36		+.013302		-1.000530		
	3	+.05		-.325339		-1.308220		
63	1	-.00	+.0967	-.097509	+.163267	-1.028435	-4.770116	+.062373
	2	-.40		-.659155		-1.654673		
	3	+.69		+.915915		-2.087008		
64	1	+.13	+.0567	+.073695	+.247951	-1.016264	-13.818279	+.038756
	2	-.69		-1.057383		-7.938848		
	3	+.73		+1.231645		-4.863167		
65	1	-.39	-.3200	-.070344	+.005018	-1.014820	-3.174239	-.318419
	2	-.13		+.197115		-1.1[illegible]5161		
	3	-.44		-.121753		-1.044258		
66	1	-.12	+.2267	-.394067	-.022492	-1.447201	-3.714500	+.232755
	2	+.51		+.308021		-1.255198		
	3	+.29		+.063554		-1.012101		
67	1	-.16	+.0267	-.193442	+.000660	-1.110956	-3.230548	+.026496
	2	+.22		+.200802		-1.119458		
	3	+.02		-.006700		-1.000134		
68	1	-.51	-.3833	-.129081	+.003074	-1.049717	-3.165866	-.382029
	2	-.44		-.057185		-1.009800		
	3	-.20		+.189340		1.106349		
69	1	+.35	+.3133	+.036749	-.001929	-1.004049	-3.119208	+.321050
	2	+.43		+.118311		-1.041802		
	3	+.16		-.156989		-1.073357		
70	1	+.59	+.2533	+.379751	+.029960	-1.416284	-3.618580	+.245021
	2	+.03		-.235018		-1.162947		
	3	+.14		-.114773		-1.039349		

Sample No	i	x_i	$m_1 = \bar{x}$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i-m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3} \frac{x_i - m}{1-(x_i-m)^2}$	$\frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3} \frac{1+(x_i-m)^2}{[1-(x_i-m)^2]^2}$	m_2 $\frac{\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
71	1 2 3	-.19 -.38 -.12	-.2300	+ .040064 - .153452 + .111347	- .002041	- 1.004812 - 1.070113 - 1.037044	- 3.111969	-.230656
72	1 2 3	-.45 -.16 +.11	-.1667	- .307638 + .007000 + .300020	- .000618	- 1.276344 - 1.000147 - 1.263129	- 3.539620	-.167175
73	1 2 3	+.71 -.48 -.07	+.0533	+1.155256 - .745580 - .125203	+ .284473	- 4.424677 - 2.508265 - 1.046789	- 7.979731	+.017651
74	1 2 3	-.33 +.15 -.17	-.1167	- .223467 + .287122 - .053451	+ .010204	- 1.147540 - 1.251454 - 1.008563	- 3.407557	-.113705
75	1 2 3	-.19 -.39 +.23	-.1167	- .073695 - .295361 + .394067	+ .025011	- 1.016264 - 1.255198 - 1.447201	- 2.718663	-.109974
76	1 2 3	-.01 -.73 -.27	- .3367	+ .365736 - .465270 + .066998	- .057547	- 1.387011 - 1.615943 - 1.013446	- 4.016400	-.351028
77	1 2 3	-.63 -.09 +.53	- .0683	- .820607 - .021710 + .931877	+ .089560	- 2.707683 - 1.001413 - 3.294334	- 7.103430	-.055692
78	1 2 3	-.04 +.47 -.61	- .0600	+ .020008 + .737032 - .815850	- .058810	- 1.001200 - 2.477060 - 2.788101	- 6.266361	-.069385
79	1 2 3	+.39 +.38 +.10	+.1900	+ .208333 + .19[illegible]15 - .090734	+ .314714	- 1.128472 - 1.115161 - 1.024631	- 3.268264	+.093706
80	1 2 3	-.77 -.16 +.15	- .2600	- .689282 + .101010 + .492847	- .095425	- 2.301754 - 1.030507 - 1.687865	- 5.020126	-.279008
81	1 2 3	-.03 -.01 -.64	- .2267	+ .204504 + .227262 - .498612	- .066846	- 1.123849 - 1.152521 - 1.703355	- 3.979725	-.243397
82	1 2 3	+.21 +.71 +.49	+ .4700	- .278850 + .254668 + .020008	- .004174	- 1.228016 - 1.190833 - 1.001200	- 3.420049	+.364611
83	1 2 3	-.26 -.28 -.08	- .2067	- .053451 - .073695 + .128767	+ .001621	- 1.008563 - 1.016264 - 1.049476	- 3.074303	-.206173
84	1 2 3	-.16 +.23 -.35	- .0933	- .066998 + .361036 - .274808	+ .019230	- 1.013446 - 1.057853 - 1.221582	- 3.292881	-.087460
85	1 2 3	-.31 +.06 +.52	+ .0900	- .476190 - .030027 + .527542	+ .021325	- 1.643990 - 1.002704 - 1.783445	- 4.430139	+.085186

Sample No.	i	x_i	$m_1 = x$ $\frac{1}{3}\sum_{i=1}^{3} x_i$	$\frac{x_i - m}{1-(x_i - m)^2}$	$\frac{\partial L}{\partial m}$ $\sum_{i=1}^{3} \frac{x_i - m}{1-(x_i - m)^2}$	$-\frac{1+(x_i - m)^2}{[1-(x_i - m)^2]^2}$	$\frac{\partial^2 L}{\partial m^2}$ $\sum_{i=1}^{3} - \frac{1+(x_i - m)^2}{[1-(x_i - m)^2]^2}$	$m_2 \cdot$ $\frac{\frac{\partial^2 L}{\partial m^2} m_1 - \frac{\partial L}{\partial m}}{\partial^2 L/\partial m^2}$
86	1	+.36	+.1233	+ .250748	- .108436	- 1.185102	- 3.533041	+.153992
	2	+.36		+ .250748		- 1.185102		
	3	-.35		- .609932		- 1.162837		
87	1	+.56	-.1567	+1.473657	+ .626575	- 6.399500	- 9.508069	-.090801
	2	-.60		- .551721		- 1.853371		
	3	-.43		- .295361		- 1.255198		
88	1	-.45	-.1833	- .287122	+ .038693	- 1.241454	- 3.814450	-.173156
	2	-.29		- .107928		- 1.034813		
	3	+.19		+ .433743		- 1.538183		
89	1	-.09	-.0800	- .010001	+ .005768	- 1.000300	- 3.710134	-.078445
	2	-.39		- .341851		- 1.341559		
	3	+.24		+ .357620		- 1.368275		
90	1	-.09	-.2500	+ .164203	- .002885	- 1.080198	- 3.198063	-.250902
	2	-.44		- .197115		- 1.115161		
	3	-.22		+ .030027		- 1.002704		
91	1	+.24	+.0800	+ .164203	+ .001397	- 1.080198	- 3.142172	+.079555
	2	+.06		- .020008		- 1.001200		
	3	-.06		- .142798		- 1.060774		
92	1	+.05	-.0867	+ .139303	- .034122	- 1.057853	- 3.632231	-.096094
	2	+.12		+ .215139		- 1.137879		
	3	-.43		- .389164		- 1.436499		
93	1	-.10	+.0833	- .189672	+ .007625	- 1.106718	- 3.300383	+.080990
	2	+.03		- .053451		- 1.008563		
	3	+.32		+ .250748		- 1.185102		
94	1	-.08	+.1233	- .212064	- .005763	- 1.133055	- 3.204354	+.125098
	2	+.26		+ .139303		- 1.057853		
	3	+.19		+ .066998		- 1.013446		
95	1	-.40	-.2233	- .183494	+ .143083	- 1.098764	- 4.961730	-.194463
	2	+.30		+ .720642		- 2.415765		
	3	-.57		- .394067		- 1.447201		
96	1	+.01	-.0700	+ .080515	+ .000291	- 1.019406	- 3.031471	-.069904
	2	-.13		- .060216		- 1.010865		
	3	-.09		- .020008		- 1.001200		
97	1	-.04	+.0433	- .083882	+ .009877	- 1.021059	- 3.279518	+.040288
	2	+.28		+ .250748		- 1.185102		
	3	-.11		- .156989		- 1.073357		
98	1	-.21	-.3067	+ .097612	- .018516	- 1.028495	- 3.240918	-.312413
	2	-.18		+ .128767		- 1.049476		
	3	-.53		- .235018		- 1.162947		
99	1	-.02	-.0700	+ .050125	+ .000000	- 1.007531	- 2.015062	-.070000
	2	-.12		- .050125		- 1.007531		
	3	-.07		+ .000000		- 0.000000		
100	1	-.09	-.0600	- .030027	+ .000000	- 1.032758	- 2.065516	-.060000
	2	-.06		+ .000000		- 0.000000		
	3	-.03		+ .030027		- 1.032758		

In order to analyze the results of the calculations on the preceding ten pages it is necessary that we find the theoretical variance for $\bar{x}$ and $\hat{m}$ from the formulae derived for this purpose lier in this paper.

From (7) page 90, we get after setting $a=1$, $p=2$, and $n=3$,

$$\sigma_{\bar{x}}^2 = \frac{1}{21} = .047619 \tag{10}$$

and from (8) page 90 after similar substitutions

$$\sigma_{\hat{m}}^2 = \frac{1}{30} = .033333 \tag{11}$$

From (9) page 90 we get for the efficiency of the mean when

$$p = 2 \quad E = .70 \tag{12}$$

Now by actual calculation from the ungrouped data the mean square deviation from zero which we shall designate as

$$\frac{\Sigma \bar{x}^2}{N} = .048611 \tag{13}$$

and

$$\frac{\Sigma m_2^2}{N} = .047612 \tag{14}$$

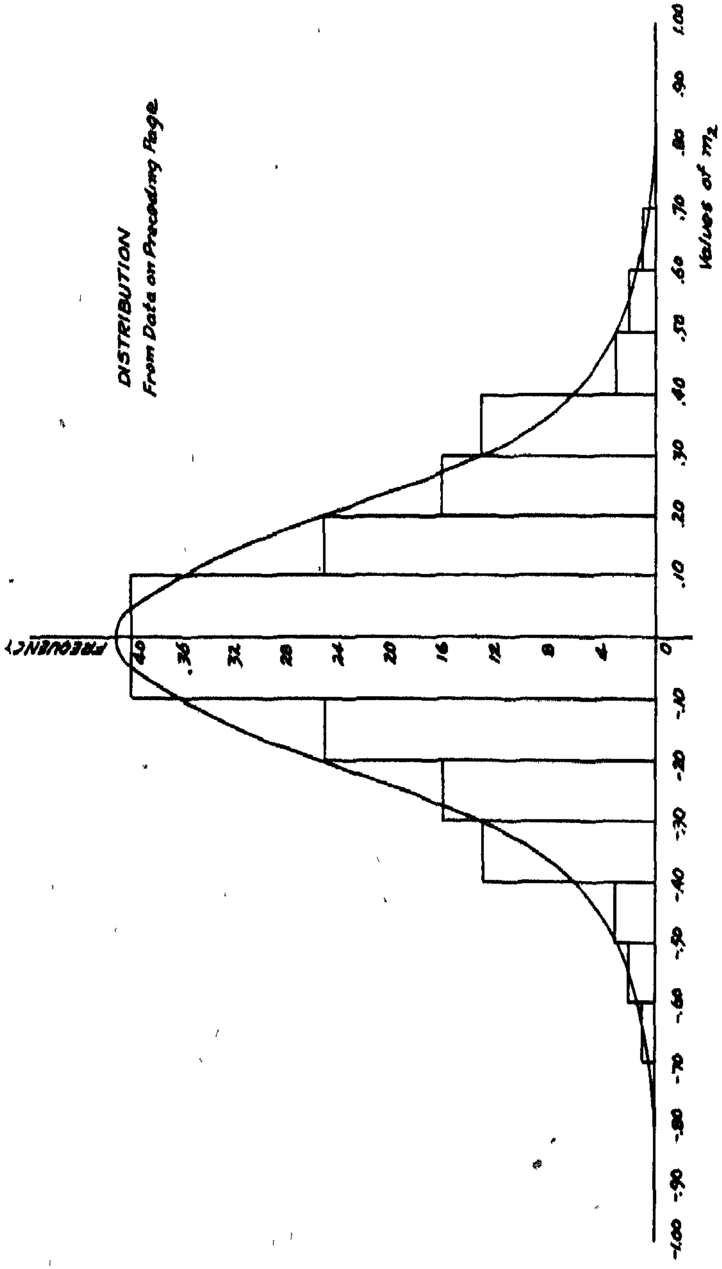
Comparing (10) and (13) it is evident that such a difference can be said to be well within the limits of random sampling. The difference between (11) and (14) is of such magnitude that we can not say that it might be expected in the course of random sampling. Now since m_2 is our estimate of $\hat{m}$ it is evident that either a single approximation by Newton's method is not adequate to give the best results or the approximation,

$$\sigma_{\hat{m}}^2 = \frac{a^2(p-1)}{np(2p+1)},$$

to the variance is not valid in the case of small samples. That the latter seems to be the case the writer firmly believes. The reason for this belief lies in the fact that in a subsequent case a sample of 3 was examined by Newton's method, and starting with $m_1 \doteq \bar{x} = -.07$ the values $m_2 = -.066879$ and $m_3 = -.066791$ were obtained. There is not sufficient improvement here to cause one to suppose that by taking a third approximation we would obtain a variance in keeping with the one derived from theory. Also considering the mean square deviation from zero for $\bar{x}$ and m_2 it seems that the gain in accuracy to be expected from the use of the method of maximum likelihood solution instead of the arithmetic mean is not sufficiently great, in the case of samples of three, to warrant the additional labor involved in calculation. We must be sure, however, that in using the arithmetic mean we are using an approximation to m which complies with the qualifications given on pages 95 and 96. A graph of the distribution as found from the calculations is given on page 106. The histogram represents the grouped data while the smoothed curve is a rough approximation to the actual form of the distribution.

Histogram Data

	m_2	+	-	Totals
	0.00	2	2	0
.01	-.10	16	20	40
.11	.20	8	17	25
.21	.30	4	12	16
.31	.40	4	9	13
.41	.50	1	2	3
.51	.60	1	1	2
.61	.70	0	1	1
.71	.80	0	0	0
.81	.90	0	0	0
.91	1.00	0	0	0
		36	64	100


DISTRIBUTION
From Data on Preceding Page
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1.00
Values of m_2

The totals have been used in the histogram which has been forced to be symmetrical so as to give the effect of a sample of twice the size.

The tabulation of the histogram data draws attention to the great excess of samples having negative values for $\bar{x}$ and m_2. This has caused the writer no little concern. In examining the signs of the observations we note that there are 183 – and only 117 + values. Assuming + and – values to be equally likely this gives a deviation from 150 of 33 or 3.81 times its standard error which is incredible. It seems therefore that Tippett's numbers are not random in this respect, and that it perhaps would have been better to toss a coin to determine the signs.

As a final check and in an effort to place the type of the distribution the value of β_2 has been calculated, and found to be

$$\beta_2 = 3.056$$

β_1 was not calculated as the excess of negative signs would lead to an erroneous value. There is every reason to believe that β_1 should be zero. These facts suggest that the curve is very near to the normal curve, but perhaps slightly more leptokurtic. But, why, if this is the case, there is not better agreement between (11) and (14) page 104 the writer is unable at the present to state.

The analytical conditions that a function of x may have a minimum are that the first derivative of the function with respect to x be zero and that the second derivative of the function with respect to x be positive. If, as is customary, we denote the first and second derivatives of $U(x)$ by $U'(x)$ and $U''(x)$ respectively, and those of $R(x)$ by $R'(x)$ and $R''(x)$ respectively, we find on differentiating

$$U'(x)=\frac{xR'(x)-C-R(x)}{x^2},$$

$$U''(x)=\frac{x^2R''(x)-2xR'(x)+2R(x)+2C}{x^3}=\frac{R''(x)-2U'(x)}{x},$$

from which it is evident that the life-time of an asset must satisfy the two conditions

$$xR'(x)-C-R(x)=0, \qquad R''(x)>0 \tag{2}$$

In short, the life-time of an asset is given by that root of the equation $xR'(x)-R(x)=C$ which will make $R''(x)>0$.

For example, let us suppose that the repair function is given by the equation $R(x)=ax^2+bx+c$. Then $R'(x)=2ax+b$, $R''(x)=2a$, and the conditions (2) reduce to

$$ax^2=C+c, \qquad a>0$$

The life-time of the asset is therefore equal to $\sqrt{(C+c)/a}$ provided the coefficient a is positive. It is interesting to observe that x is independent of the constant b.

3. In the preceding discussion no allowance was made for the salvage value of the asset. Let us denote the scrap-value of the asset after x years by $S(x)$, then the average yearly de-

preciation, interest again not considered, is

$$U(x) = \frac{C + R(x) - S(x)}{x}, \tag{3}$$

and the conditions which will make $U(x)$ a minimum are

$$x\left[R'(x) - S'(x)\right] - \left[C + R(x) - S(x)\right] = 0, \quad R''(x) > S''(x). \tag{4}$$

If the scrap-value is a constant, both $S'(x)$ and $S''(x)$ vanish, and the life-time of the asset is determined by

$$xR'(x) - C - R(x) + S(x) = 0, \quad R''(x) > 0. \tag{5}$$

The conditions (2) include (4) if we replace $R(x)$ by $R(x) - S(x)$, that is, if in the outset we diminish the repair function by the salvage value at time x ; to include (5) it is sufficient to replace C, the original cost of the asset, by $C - S$, the difference between original cost and scrap value. With these modifications we may treat (2) as representing the general case.

4. To avoid any possible confusion, let us denote by $T(x)$ the total outlay to be recovered by uniform annual charges to production during the life-time of the asset. Taking account of the residual value $S(x)$ we see that

$$T(x) = C + R(x) - S(x), \qquad T'(x) = R'(x) - S'(x) \tag{6}$$

and (3) and (4) take the simpler forms

$$U(x) = T(x)/x, \qquad xT'(x) - T(x) = 0, \quad R''(x) > S''(x). \tag{7}$$

From the first and second of the equations (7) follows:

$$T'(x) = U(x) \tag{8}$$

which may be appropriately called the life-equation of an asset since its solution yields the life-time of the asset as defined in 1.

5. When the repair function and the salvage function are known, the real roots of the life-equation may be found either by direct methods or by methods of approximation. However, in the great majority of cases which occur in practice the value of $T(x)$ is given only empirically, from the recorded experience relating to the asset in question, and the data available may not lend itself to analytical treatment. In all such cases the life-time of the asset may be determined approximately by means of the following simple graphic method.

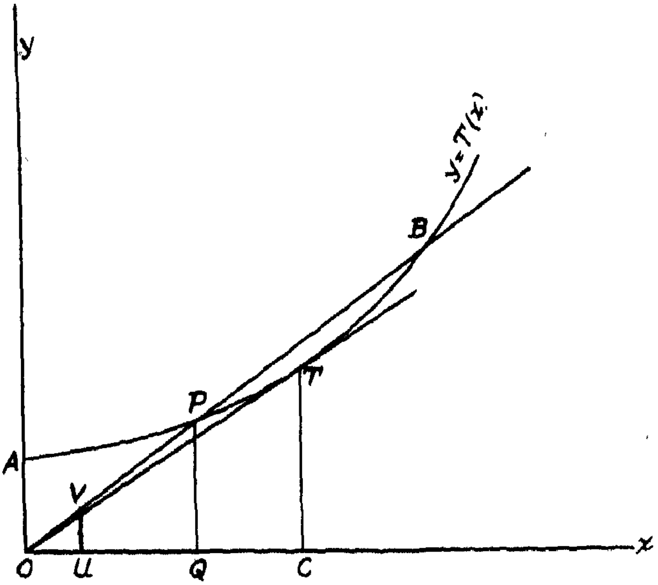

FIGURE 1.

Let AB (Fig. 1) represent the graph of the equation $y=T(x)$, constructed in Cartesian coordinates. We shall call it the total outlay graph, because the ordinate y of any point (x, y) on this graph represents the total outlay during the period of x years if the asset were scrapped at the end of this period. The straight line, OP, joining the origin O to any point P on AB, we shall call the uniform charge to production graph. It enables us to determine at sight the aggregate amount that must be charged to production during any given period of time in order to recover the total outlay OP for the time x on the basis of uniform distribution over the entire period x. If x is expressed in years, the ordinate UV, of the point on OP whose abscissa is unity, will represent the uniform charge to production per year, which is required to recover the total outlay for x years.

Now it is obvious that this unit charge UV will vary with the slope of the line OP. It will be least when the slope is least, that is to say, when the point P is such that the line OP is tangent to the total outlay graph. The abscissa, OC, of the point of contact, T, is then the life-time of the asset under consideration.

To determine the life-time of an asset, interest considerations being disregarded, we need therefore only construct the total outlay graph AB, then draw the tangent OT, and finally measure the abscissa of the point of contact T

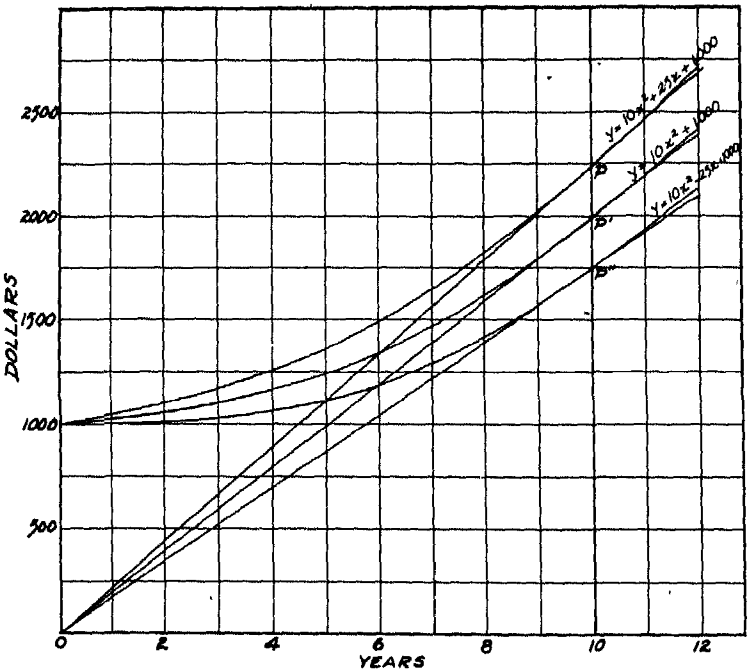

FIGURE 2

(Fig. 2) shows the construction when $T(x)$ has the forms

$$10x^2 + 25x + 1000,$$
$$10x^2 + 1000, \text{ and}$$
$$10x^2 - 25x + 1000$$

respectively. In each case the life-time is found to be 10 years, which verifies the theoretical conclusion of 2: that the life-time is independent of the coefficient of x.

We have seen from graphical considerations that the uniform charge to production will be a minimum when its graph is tangent to the total outlay graph. This condition is precisely the condition expressed by equation (8), which asserts that when x is the life-time of the asset $T'(x)$, the slope of the tangent to the total outlay graph, is numerically the same as the yearly charge to production.

6. We now come to consider the problem of finding the life-time of an asset when interest at a specified rate is to be taken into account. In this case, the various items that make up the total outlay, as well as the component charges to production, must be replaced by their present values at some arbitrarily chosen epoch, as say, the epoch zero.

Let us attempt an analytical solution of the problem. Let Δt represent a small interval of time. The outlay during the interval from t to $t+\Delta t$ is $T(t+\Delta t)-T(t)$. If the specified rate of interest is, i, and if we represent the discount factor by $1/(1+i)$ the conventional symbol v, then the present value of

$$T(t+\Delta t)-T(t)$$

at the epoch 0 has some value between $[T(t+\Delta t)-T(t)]v^t$ and $[T(t+\Delta t)-T(t)]v^{t+\Delta t}$, let us say

$$[T(t+\Delta t)-T(t)]v^{t+\theta\Delta t}$$

where θ has some value between 0 and 1. The total outlay during the time t, evaluated for the epoch 0, is therefore,

(9) $$C+\Sigma[T(t+\Delta t)-T(t)]v^{t+\theta\Delta t}$$

the sum extending over all the time intervals between 0 and t.

Now $v^t > v^{t+\theta\cdot\Delta t} > v^{t+\theta_M\cdot\Delta t}$ where θ_M is the greatest among all the fractions θ. We have, therefore,

$$\begin{aligned}(10)\qquad \left[T(t+\Delta t)-T(t)\right]v^t &> \left[T(t+\Delta t)-T(t)\right]v^{t+\theta\cdot\Delta t}\\ &> \left[T(t+\Delta t)-T(t)\right]v^{t+\theta_M\cdot\Delta t}\end{aligned}$$

If the intervals Δt are all equal and their number n, then $\Delta t = t/n$, and as n is increased indefinitely Δt approaches 0 Then $t+\theta_M\cdot\Delta t$ approaches t, and we see from (10) that (9) must have the same limit as

$$(11)\qquad C+\sum\left[T(t+\Delta t)-T(t)\right]v^t$$

To determine this limit we write

$$T(t+\Delta t)-T(t) = \frac{T(t+\Delta t)-T(t)}{\Delta t}\Delta t$$

where the first factor on the right represents the difference quotient which approaches $T'(t)$ as a limit as Δt approaches 0 as a limit. With this relation introduced into (11), we obtain for the present value at the epoch 0 of all the increments of outlay during the time t, the intervals of time being infinitesimal,

$$\begin{aligned}(12)\qquad \overline{T}(t) &= C+\underset{\Delta t\to 0}{\text{Limit}}\left[\sum \frac{T(t+\Delta t)-T(t)}{\Delta t}\cdot v^t\Delta t\right] =\\ &C+\int_0^t v^t\cdot T'(t)\cdot dt.\end{aligned}$$

In a like manner we may derive an expression for $\overline{D}(t)$, the limit of the sum of the present values at epoch 0 of all the charges to production during the time t apportioned at some uniform rate U to each of the intervals Δt . The charge apportioned to the interval from t to $t+\Delta t$ is $U.\Delta t$, its present value at epoch O is $U.\,v^{t+\theta\Delta t}.\Delta t$. The present value of the sum of these amounts for all the intervals Δt between O and t is

$$\Sigma\, U.\, v^{t+\theta.\Delta t}.\Delta t$$

which for infinitesimal values of Δt has the same limit as $\Sigma\, U.\, v^{t}\, \Delta t$, so that finally

$$\bar{D}(t) = \underset{\Delta t \to O}{\text{Limit}}\, \Sigma\, U.\, v^{t}.\Delta t = U\int_{0}^{t} v^{t}\, dt. \tag{13}$$

Let $U(x)$ be the value which must be assigned to U in order to recover $\bar{T}(x)$, the total outlay for x years through a uniform charge to production, interest considered. Then $\bar{D}(x)$ must equal $\bar{T}(x)$, that is,

$$U(x).\int_{0}^{x} v^{t}.dt = C + \int_{0}^{x} v^{t}.\, T'(t).dt,$$

from which

$$U(x) = \frac{C + \int_{0}^{x} v^{t}.\, T'(t).dt}{\int_{0}^{x} v^{t}.dt} \tag{14}$$

The life-time of the asset is that value of x in (14) which will make $U(x)$ a minimum. The derivative of $U(x)$ with respect to x must therefore vanish. Differentiating (14) with respect to x and setting the result equal to O, we find

$$v^{x}.\, T'(x)\int_{0}^{x} v^{t}.dt - \left[C + \int_{0}^{x} v^{t}.\, T'(t).\, dt\right] v^{x} = O,$$

from which

$$T'(x) = \frac{C + \int_{0}^{x} v^{t}.\, T'(t).dt}{\int_{0}^{x} v^{t}.dt}\,, \tag{15}$$

which is the life-equation of the asset, interest considered.

7. In deriving equation (13) we apportioned the charges to production for an interval Δt and found the sum of the present values at epoch O. $\bar{D}(t)$ is the limiting value of this sum as the intervals Δt are indefinitely diminished. If, as is customary, no charge is made to production until the end of the year, this single charge will be the aggregate amount of the constituent portions for the separate intervals Δt, accumulated with interest to the end of the year. The charge for the interval from t to $t+\Delta t$ is $U\Delta t$, its amount at rate i to the end of the year is $U(1+i)^t.\Delta t$, where t is the time to the end of the year, and the equivalent single charge at the end of the year is

$$\bar{U} = \underset{\Delta t\to 0}{limit}\sum U.(1+i)^t.\Delta t = U.\underset{\Delta t\to 0}{limit}\sum(1+i)^t.\Delta t$$

$$(16)\qquad = U\int_0^i (1+i)^t.dt = \frac{iU}{\log(1+i)}.$$

8. As an example let us again take $T(t)=at^2+bt+c$, then

$$T'(t)=2at+b \quad \int_0^t v^t dt = v^t/\log v\Big|_0^t = (v^t-1)/\log v,$$

$$\int_0^t v^t.T(t).dt = \int_0^t v^t(2at+b)dt$$

$$= \left[2atv^t+b(v^t-1)\right]/\log v - 2a(v^t-1)/(\log v)^2$$

(17)

$$\bar{T}(t) = c+\left[2atv^t+b(v^t-1)\right]/\log v - 2a(v^t-1)/(\log v)^2,$$

$$(18)\qquad \bar{D}(t) = U.(v^t-1)/\log v,$$

and (15) reduces to

$$(19) \qquad v^x - \log v^x = 1 + \frac{c(\log v)^2}{2a}.$$

While the life-equation (19) cannot be solved algebraically, it is evident that an approximate solution for could be obtained from a list of tabulated values of the function $v^x - \log v^x$. When such a table is not available, an approximate solution to any desired degree of accuracy may be obtained as follows:

We may write for v, $e^{\log v}$ where e is the base of the natural system of logarithms. (19) then takes the form

$$(20) \qquad e^{x \log v} - x \log v = 1 + c(\log v)^2/2a.$$

On expanding the first term of this equation into a power series in x, and simplifying the result, we have

$$x^2 + x^3(\log v)/3 + x^4(\log v)^2/12 + x(\log v)^3/60 + \cdots = c/a,$$

whence

$$x^2 = \frac{c/a}{1 + x(\log v)/3 + x^2(\log v)^2/12 + x^3(\log v)^3/60 + \cdots}.$$

Now for all ordinary rates of interest log v is necessarily very small, so that if we denote successive approximations of x by x_1, x_2, x_3, etc.,

$$x_1 = (c/a)^{1/2}, \quad x = \left[\frac{c/a}{1 + x_1(\log v)/3}\right]^{1/2},$$

$$x_3 = \left[\frac{c/a}{1 + x_2(\log v)/3 + x_2^2(\log v)^2/12}\right]^{1/2}$$

$$x_4 = \left[\frac{c/a}{1 + x_3(\log v)/3 + x_3^2(\log v)^2 12 + x_3^3(\log v)^3/60}\right]^{1/2}$$

, etc.

Let us take the special case, previously considered in 5., when $c = 1000$, $b = 0$, $a = 10$, and let the assumed rate of interest be 6 percent. Then

$$\log v = -0.058\,269, \quad (\log v)^2 = 0.003\,395,$$

$$1/\log v = -17\,161.788, \quad 1/(\log v)^2 = 294.526\,967,$$

and we find

$$T(t) = 10t^2 + 1000,$$

$$\bar{T}(t) = 1000 - 343\,24\,t v^t - 5890\,54(v^t - 1),$$

$$\bar{D}(t) = 17.162\,U(1 - v^t),$$

and the life-equation is

$$v^x - \log v^x = 1\,169\,764$$

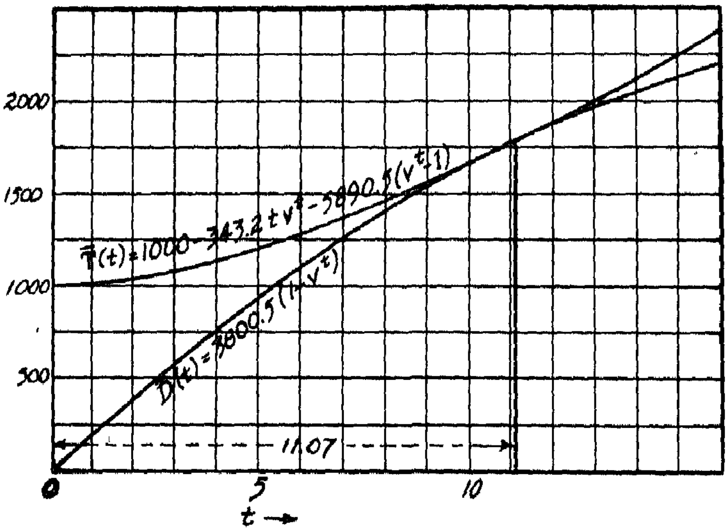

FIGURE 3

The first four successive approximations for x give

$x_1 = 10$, $x_2 = 11.14$, $x_3 = 11.05$

$x_4 = 11.07$

The value $x = 11.07$ substituted in (14) and (16) give us

$$U(x) = 221.45 \quad \text{and} \quad \bar{U}(x) = 228.03$$

This value of $U(x)$ substituted for U in the expression for $\bar{D}(t)$ gives

$$\bar{D}(t) = 3800.5(1 - v^{t}),$$

which represents the present value at epoch O of the aggregate momentary charges during a period t at a rate such as to recover the total outlay 11.07 years, the theoretical life-time of the asset. The momentary rate is 221.45 per year, the equivalent single charge to production at the end of each year is 228.03.

(Fig. 3) shows the graphs of the two equations.

$$\bar{T}(t) = 1000 - 343.24\,t\,v^{t} - 5890$$

and

$$\bar{D}(t) = 3800.5(1 - v^{t}).$$

The abscissa of the common ordinate of the two curves represents the life-time of the asset.

9. It appears from (Fig. 3) that at the point common to the two graphs, the graphs have a common tangent as well as a common ordinate. To see whether or not this is a general property let us trace the changes in the total outlay and total charge to production functions when interest is taken into account.

In the first place it is evident that the increments of the ordinates of both of the graphs in (Fig. 1) must be replaced by their present values at the chosen epoch. If this epoch is the effect in question will be to shorten progressively the ordinates of both graphs. The charge to production graph will then be no longer a straight line but some convex curve, while the total outlay graph will go over into another graph which is less concave than the original graph. But both graphs will continue to rise indefinitely as we proceed from left to right because the increments of their ordinates, while decreasing indefinitely remain positive.

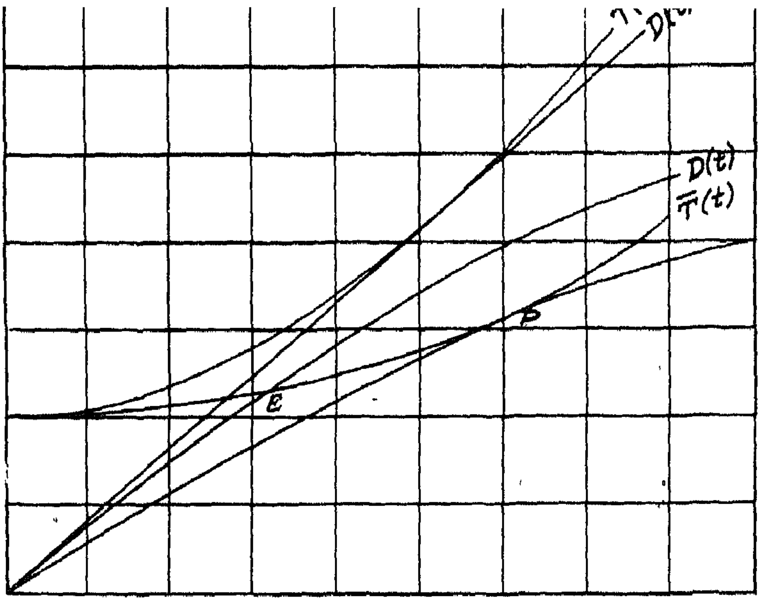

FIG. 4

In (Fig. 4) let $T(t)$ and $D(t)$ represent respectively the total outlay graph and the charge to production graph, interest disregarded, $\bar{T}(t)$ the total outlay graph, interest considered, and $\bar{D}(t)$ the charge to production graph, interest considered, through any point E on $\bar{T}(t)$. The ordinates on $\bar{D}(t)$ represent the present values at epoch O of the momentary charges to production during time t at a rate such as to recover the entire outlay during the time corresponding to the abscissa of the point E. This rate is measured by the initial slope of $\bar{D}(t)$, the slope of $\bar{D}(t)$ when $t=O$.

Let us follow the changes in this slope for the various positions of the point E as it moves along $\bar{T}(t)$ from left to right. It is evident that this slope at first decreases, also that it cannot keep on decreasing indefinitely, it is therefore plausible that it will ultimately increase, reaching a minimum value at the point P where the $\bar{T}(t)$ curve and the $\bar{D}(t)$ curve have a common tangent. The abscissa of the point of contact, P, is then the lifetime of the asset under discussion.

10. The foregoing considerations, however plausible, are open to objections, because we have reasoned from graphs resulting from the assumption of a special law governing the repair function. Different assumptions might give rise to essentially different graphs. We shall, therefore establish the conclusions above arrived at, by an analytical proof, which is independent of any assumptions regarding the nature of the outlay function. We shall prove the

Theorem: If the rate U of a uniform charge to production curve is a minimum, this curve is tangent to the corresponding total outlay curve, and the abscissa of the point of contact represents the life-time of the asset. Conversely,

If a uniform charge to production curve is tangent to the corresponding total outlay curve, U is a minimum.

To prove this theorem, let $y = \bar{T}(t)$ be the equation of the total outlay curve, $y = \bar{D}(t) = U\int_0^t v^t dt$, the equation of the uniform charge to production curve, and x the abscissa of a point common to the two curves.

Then $\bar{T}(x) = \bar{D}(x) = U\int_0^x v^t dt$ from which

$$U = \bar{T}(x)/\int_0^x v^t dt. \tag{21}$$

Since by hypothesis U is a minimum, its derivative with respect to x must vanish, that is

$$\bar{T}'(x)\int_0^x v^t dt - v^x \bar{T}(x) = 0 \tag{22}$$

From (22) and (21) follows

$$\bar{T}'(x) = v^x \bar{T}(x)/\int_0^x v^t dt = v^x U = \bar{D}'(x). \tag{23}$$

This shows that at the point common to the two curves their slopes are equal, they have therefore a common tangent, and since

U has a minimum value, x must represent the life-time of the asset.

To prove the converse theorem we observe that if the two curves have a common tangent at the point $t=x$,

$$(24) \qquad \bar{T}(x) = \bar{D}(x) = U\int_0^x v^t dt$$

and

$$(25) \qquad \bar{T}'(x) = \bar{D}'(x) = v^x U$$

Substituting the value of U from (24) in (25) we find

$$\bar{T}(x) = v^x \bar{T}(x) / \int_0^x v^t dt$$

from which

$$\bar{T}(x)\int_0^x v^t dt - v^x \bar{T}(x) = 0.$$

But by (22) this is precisely the condition that U is a minimum.

11. In most cases which arise in practice the analytical method of finding the life-equation of an asset fails owing to the empirical character of the outlay function. The question suggests itself whether a graphic method, similar to that employed in the simpler case treated in 7, can be devised, which will yield an approximate solution of the problem. The theorems of the preceding article offer the key to such a method.

Let us suppose that the total outlay graph has been constructed on a convenient scale, the scale depending on the magnitude of the quantities involved. Every point on this curve determines a definite uniform charge to production curve. We seek that particular one of these curves which is tangent to the total outlay graph. The abscissa of the point of contact would

THE SIMULTANEOUS DISTRIBUTION OF MEAN AND STANDARD DEVIATION IN SMALL SAMPLES

By ALLEN T. CRAIG

1. Introduction. If samples of n items are selected at random from a normal universe, it is well known that the arithmetic mean $\bar{x}$ and standard deviation s computed from samples are independent in the probability sense and that the simultaneous frequency distribution is

$$F(\bar{x},s)=Cs^{n-2}e^{-\frac{ns^2+n\bar{x}^2}{2\sigma^2}},$$

If, however, the parent population is other than the normal type, there appears to be little known regarding the form of $F(\bar{x}, s)$. In the present paper, we propose to determine the simultaneous frequency function of the arithmetic mean and standard deviation in samples of small numbers of items selected at random from a rather arbitrary universe. For convenience, we shall classify frequency distributions according as the range of the independent variable is $(-\infty, \infty), (0, \infty)$ or $(0, a), a>0$. We shall further assume that the total area under the distribution function is unity.

2. The simultaneous distribution of $\bar{x}$ and s in samples of $n=2$. Let $f(x), -\infty<x<\infty$ be the frequency function of the variable x. Let x_1 and x_2 be two independently observed values of x. write

$$x_1+x_2=2\bar{x}$$
$$x_1^2+x_2^2=2s^2+2\bar{x}^2.$$

We seek the function $F(\bar{x}, s)$ such that $F(\bar{x}, s)\,d\bar{x}\,ds$ is, to within infinitesimals of higher order, the probability of the simultaneous occurrence of $\bar{x}$ in $(\bar{x}, \bar{x}+d\bar{x})$ and s in $(s, s+ds)$. For $\bar{x}$ and s assigned, x_1 may have either value $\bar{x}-s$ or $\bar{x}+s$ and x_2 is uniquely determined by $x_2 = 2\bar{x}-x_1$.

Thus

$$F(\bar{x},s)\,d\bar{x}\,ds = f(\bar{x}-s)f(2\bar{x}-x_1)\,dx_1\,dx_2$$
$$+f(\bar{x}+s)f(2\bar{x}-x_1)\,dx_1\,dx_2 .$$

Since $dx_1\,dx_2 = 2\,d\bar{x}\,ds$ we have

$$(1) \qquad F(\bar{x},s) = 4f(\bar{x}-s)\,f(\bar{x}+s).$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $s \leq \bar{x}$. Thus (1) is valid for this type of frequency function but the surface is limited by the x-axis and the line $s = \bar{x}$

If $f(x)$ is defined on the interval $(0, a)$, we note, for $\bar{x}$ assigned on $(0, a/2)$, that $s \leq \bar{x}$; and, for $\bar{x}$ assigned on $(a/2, a)$ that $s \leq a-\bar{x}$. Accordingly, for this kind of frequency function, (1) is valid but the surface is limited by the x-axis and the lines $s = \bar{x}$, $s = a-\bar{x}$.

As simple illustrations, let us find the correlation surface for the mean and standard deviation of samples of two items drawn from distributions of various types.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}, \qquad -\infty < x < \infty .$$

Then

$$F(\bar{x},S)=\frac{2}{\sigma^2\pi}e^{-\frac{s^2+\bar{x}^2}{\sigma^2}},$$

the well known result.

Example 2. Let

$$f(x)=e^{-x}, \qquad 0\le x<\infty.$$

Then

$$F(\bar{x},S)=4e^{-2\bar{x}}$$

over the open region of the $\bar{x}s$-plane bounded by the $\bar{x}$-axis and the line $s=\bar{x}$.

Example 3. Let

$$f(x)=\frac{1}{a}, \qquad 0\le x\le a.$$

Then

$$F(\bar{x},S)=\frac{4}{a^2},$$

over the region of the $\bar{x}s$-plane bounded by the isosceles triangle with sides $s=0$, $s=\bar{x}$ and $s=a-\bar{x}$. With a uniform distribution proportional to $4/a^2$ over this triangle, it follows incidentally from very elementary geometry that the marginal totals of the distribution of $\bar{x}$ are given by the known values

$$\phi(\bar{x})=\frac{4}{a^2}\bar{x}, \qquad 0\le\bar{x}\le\frac{a}{2},$$

$$=\frac{4}{a^2}(a-\bar{x}), \qquad \frac{a}{2}\le\bar{x}\le a,$$

and that the marginal totals for the distribution of s are given by

$$\psi(s) = \frac{4}{a^2}(a-2s), \qquad 0 \leq s \leq \frac{a}{2},$$

which is the result given by Rider.[1]

3. The simultaneous distribution of $\bar{x}$ and s in samples of $n=3$. Consider first a frequency function $f(x), -\infty < x < \infty$. We have

$$x_1 + x_2 + x_3 = 3\bar{x},$$

$$x_1^2 + x_2^2 + x_3^2 = 3s^2 + 3\bar{x}^2.$$

Upon eliminating x_3, we have

$$2x_1^2 + 2x_1x_2 + 2x_2^2 - 6\bar{x}x_1 - 6\bar{x}x_2 - 3s^2 + 6\bar{x}^2 = 0.$$

From simple properties of this ellipse, it follows, for assigned $\bar{x}$ and s that x_1 may be chosen arbitrarily from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$. With x_1 assigned, x_2 must be selected with certainty as either

$$\frac{3\bar{x} - x_1 - \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2} \quad \text{or}$$

$$\frac{3\bar{x} - x_1 + \left[6s^2 - 3(x_1 - \bar{x})^2\right]^{\frac{1}{2}}}{2}$$

Finally we must have

$$x_3 = 3\bar{x} - x_1 - x_2.$$

---

[1] P. R. Rider, On the distribution of ratio of mean to standard deviation etc., Biometrika, vol. 21 (1929) pp. 124-141.

Thus

$$F(\bar{x},s)d\bar{x}\,ds = 2\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} f(x_1)f(x_2)f(x_3)dx_1dx_2dx_3.$$

From

$$\begin{aligned} x_1 &= x_1, \\ x_2 &= \frac{3\bar{x}-x_1 \pm \left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{2}, \\ x_3 &= 3\bar{x}-x_1-x_2 \end{aligned}$$

we obtain

$$dx_1dx_2dx_3 = \frac{9s}{\left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}}\,dx_1d\bar{x}\,ds$$

$$= 9s\,dx_1d\bar{x}\,ds/R$$

where

$$R \equiv \left[6s^2-3(x_1-\bar{x})^2\right]^{\frac{1}{2}}.$$

Thus

(2)

$$F(\bar{x},s) = 18s\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{1}{R}\, f(x_1)\, f\!\left(\frac{3\bar{x}-x_1+R}{2}\right) f\!\left(\frac{3\bar{x}-x_1-R}{2}\right)dx_1$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for $\bar{x}$ assigned, that $0 \le s \le \bar{x}\sqrt{2}$. Thus, the surface is limited by the $\bar{x}$-axis and the line $s = \bar{x}\sqrt{2}$. Moreover, since x_1, x_2, x_3 are non-negative, x_1 may be selected from the interval $(\bar{x} - s\sqrt{2}, x + s\sqrt{2})$ only as long as $s \le \bar{x}\sqrt{2}/2$. If

$$\bar{x}\ \bar{2}\sqrt{2} \le s \le \bar{x}\sqrt{2}.$$

then x_1 may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2}\right).$$

Accordingly, for this type of frequency function,

$$F(\bar{x},s) = 18s\int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x}-x_1+R}{2}\right) f\left(\frac{3\bar{x}-x_1-R}{2}\right) dx_1,$$

$$0 \le s \le \frac{\bar{x}\sqrt{2}}{2},$$

(2.1)

$$= 18s\left[\int_0^{\frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x}+s\sqrt{2}}\right]$$

$$\frac{1}{R} f(x_1) f\left(\frac{3\bar{x}-x_1+R}{2}\right) f\left(\frac{3\bar{x}-x_1-R}{2}\right) dx_1,$$

$$\frac{\bar{x}\sqrt{2}}{2} \le s \le \bar{x}\sqrt{2}.$$

If $f(x)$ is defined on the interval $(0, a)$, we note:

for $0 \le \bar{x} \le a/3$, $\quad 0 \le s \le \bar{x}\sqrt{2}$;

for $a/3 \le \bar{x} \le 2a/3$, $\quad 0 \le s \le \left[2\bar{x}^2 - 2a\bar{x} + \frac{2a^2}{3}\right]^{\frac{1}{2}}$;

for $2a/3 \le \bar{x} \le a$, $\quad 0 \le s \le (a-\bar{x})\sqrt{2}$.

Thus in this case, the surface is limited by the x-axis, the lines $s = \bar{x}\sqrt{2}$ and $s = (a-\bar{x})\sqrt{2}$ and the hyperbola

$$s = \left[2\bar{x}^2 - 2a\bar{x} + 2a^2/3\right]^{\frac{1}{2}}.$$

(Fig. 1.). Now x_1 may be selected from the interval $(\bar{x} - s\sqrt{2}, x + s\sqrt{2})$ as long as $s \le \bar{x}\sqrt{2}/2$ and $s \le (a-\bar{x})\sqrt{2}/2$. This holds for that part of the surface over the region bounded by OPa. For that part of the surface over the region bounded by OPU, x_1 may be selected from the intervals

$$\left(0, \frac{3\bar{x} - \left[6s^2 - 3\bar{x}^2\right]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} + \left[6s^2 - 3\bar{x}^2\right]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2}\right).$$

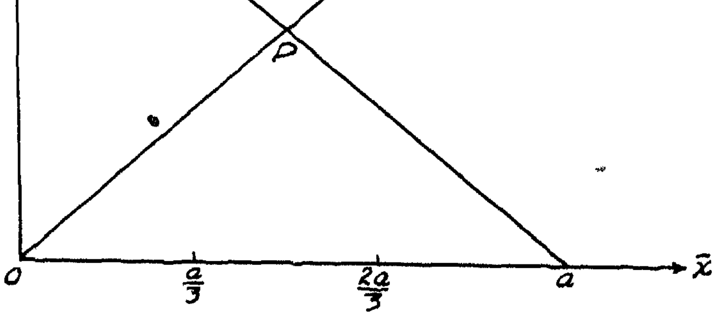

Fig. 1

It is clear that the ranges of arbitrary selection of x_1 for that part of the surface over the region bounded by PVa are

$$\left(\bar{x} - s\sqrt{2},\ \frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}\right)$$

and

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ a\ .\right)$$

Finally, we find that x_1 may be selected from the intervals

$$\left(0,\ \frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}\right),$$

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2},\ \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}\right),$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2},\ a\right)$$

for that part of the surface over the region bounded by PUV . If we adopt the notation

$$\phi = \phi(x_1, \bar{x}, s) \equiv \frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right),$$

we have

$$F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \phi\, dx_1, \tag{2.2}$$

$$= 18s\left[\int_0^{\frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}}\right] \phi\, dx_1,$$

$$= 18s\left[\int_{\bar{x} - s\sqrt{2}}^{\frac{3\bar{x} - a - [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}\right.$$

$$\left. + \int_{\frac{3\bar{x} - a + [6s^2 - 3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{a}\right] \phi\, dx_1

$$= 18s\left[\int_0^{\frac{3\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{2}}^{\frac{3\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{2}}^{a}\right]\phi\, dx_1,$$

for the parts of the surface over the regions indicated above.

In order to illustrate the theory, we shall consider a few examples.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

By (2),

$$F(\bar{x}, s) = \frac{3\sqrt{3}}{\sigma^3\sqrt{2\pi}}\, s e^{-\frac{3s^2+3\bar{x}^2}{2\sigma^2}}.$$

Example 2. Let

$$f(x) = e^{-x}, \qquad 0 \leq x < \infty.$$

By (2.1),

$$F(\bar{x}, s) = 6\sqrt{3}\pi s e^{-3\bar{x}}, \qquad 0 \leq s \leq \frac{\bar{x}\sqrt{2}}{2},$$

$$= 6\sqrt{3} s e^{-3\bar{x}}\left[\arcsin \frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \arcsin\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left. - \arcsin \frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \frac{\pi}{2}\right], \quad \frac{\bar{x}\sqrt{2}}{2} \leq s \leq \bar{x}\sqrt{2}.$$

Example 3. Let

$$f(x) = \frac{1}{a}, \qquad 0 \leq x \leq a.$$

By (2.2),

$$F(\bar{x},s) = \frac{6\sqrt{3}\,\pi s}{a^3}\text{, over } OPa,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\text{arc sin}\,\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s\,2\sqrt{2}} + \text{arc sin}\,\frac{\bar{x}}{s\sqrt{2}}\right.$$

$$\left. -\,\text{arc sin}\,\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s\,2\sqrt{2}} + \frac{\pi}{2}\right]\text{, over } OPU,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\text{arc sin}\,\frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}\right.$$

$$+\,\text{arc sin}\,\frac{a-\bar{x}}{s\sqrt{2}} - \text{arc sin}\,\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left. +\,\frac{\pi}{2}\right]\text{, over } \quad PVa,$$

$$= \frac{6\sqrt{3}s}{a^3}\left[\text{arc sin}\,\frac{\bar{x}-a-[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s\,2\sqrt{2}}\right.$$

$$+\,\text{arc sin}\,\frac{\bar{x}}{s\sqrt{2}} + \text{arc sin}\,\frac{\bar{x}-[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$+\,\text{arc sin}\,\frac{a-\bar{x}}{s\sqrt{2}}$$

$$-\,\text{arc sin}\,\frac{\bar{x}-a+[6s^2-3(a-\bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}}$$

$$\left. -\,\text{arc sin}\,\frac{\bar{x}+[6s^2-3\bar{x}^2]^{\frac{1}{2}}}{s\,2\sqrt{2}}\right]\text{, over } PVU.$$

I have succeeded in obtaining the marginal totals for s from 0 to $a\sqrt{2}/4$ by integrating $F(\bar{x},s)$ with respect to $\bar{x}$ from the boundary (Fig. 1) $s=\bar{x}\sqrt{2}$ to $s=(a-\bar{x})\sqrt{2}$ and obtain as a result the parabola which is known[1] to give the distribution of s from $s=0$ to $s=a\sqrt{6}/6$.

4. The simultaneous distribution of $\bar{x}$ and s in samples of $n=4$. We shall consider first samples of four items drawn from a universe characterized by a law of frequency $f(x)$, $-\infty<x<\infty$. Then

$$x_1+x_2+x_3+x_4=4\bar{x},$$
$$x_1^2+x_2^2+x_3^2+x_4^2=4s^2+4\bar{x}^2.$$

The elimination of x_4 yields

$$x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3-4\bar{x}x_1-4\bar{x}x_2-4\bar{x}x_3-2s^2+6\bar{x}^2=0$$

It follows from the properties of this ellipsoid that x_1 may be chosen arbitrarily from the interval $(\bar{x}-s\sqrt{3},\ \bar{x}+s\sqrt{3})$. For x_1 assigned, the region of arbitrary selection of x_2 is determined by the properties of the ellipse and is

$$\left(\frac{4\bar{x}-x_1-2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3},\ \frac{4\bar{x}-x_1+2\left[6s^2-2(x_1-\bar{x})^2\right]^{\frac{1}{2}}}{3}.\right)$$

Upon solving for x_3 in terms of x_1 and x_2 we have

$$x_3=\frac{4\bar{x}-x_1-x_2\pm\left[8s^2-8\bar{x}^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}}{2}$$

while x_4 is uniquely determined by $x_4=4\bar{x}-x_1-x_2-x_3$.
If we write

$$T\equiv\left[8s^2-8x^2+8\bar{x}x_1+8\bar{x}x_2-2x_1x_2-3x_1^2-3x_2^2\right]^{\frac{1}{2}}$$

and

$$\phi\equiv f(x_1)f(x_2)f\left(\frac{4\bar{x}-x_1-x_2+T}{2}\right)f\left(\frac{4\bar{x}-x_1-x_2-T}{2}\right)$$

---

[1] H. L. Rietz [Paper to appear presently in Biometrika].

then

$$F(\bar{x},s)=32s\int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\frac{1}{T}\phi\,dx_2\,dx_1. \tag{3}$$

The integration can be carried out in an obvious manner when $f(x)$ is the normal frequency function.

In case $f(x)$ is defined on the interval $(0,\infty)$, we note, for $\bar{x}$ assigned, that $s\le\bar{x}\sqrt{3}$. Thus the surface is limited by the $\bar{x}$-axis and the line $s=\bar{x}\sqrt{3}$ Moreover, x_1 may be selected from the interval $(\bar{x}-s\sqrt{3},\ \bar{x}+s\sqrt{3})$ with x_2 chosen as above only as long as $s\le\bar{x}\sqrt{3}/3$ If $\bar{x}\sqrt{3}/3\le s\le\bar{x}$, then x_1 may be chosen from either of the two intervals

$$\left(0,\ \frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

and

$$\left(\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3},\ \bar{x}+s\sqrt{3}\right)$$

with x_2 chosen as above; or x_1 may be selected from the interval

$$\left(\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3},\ \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}\right)$$

with x_2 taken from either

$$\left(0,\ \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}\right)$$

or

$$\left(\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2},\ \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}\right).$$

when $\bar{x}\le s\le\bar{x}\sqrt{3}$ we may have

$$0\le x_1\le 2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}$$

and

$$2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}} \leq x_1 \leq \frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}$$

with either

$$0 \leq x_2 \leq \frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}$$

or

$$\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2} \leq x_2 \leq \frac{4\bar{x}-x_1+2[6s-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Or we may have

$$\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3} \leq x_1 \leq \bar{x}+s\sqrt{3}$$

with

$$\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3} \leq x_2 \leq \frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}$$

Accordingly, for this kind of frequency function,

$$F(\bar{x},s)=32s\int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-x)^2]^{\frac{1}{2}}}{3}} \frac{1}{7}\phi \, dx_2 \, dx_1,$$

(3.1)

$$0 \leq s \leq \frac{\bar{x}\sqrt{3}}{3},$$

$$=32s\left[\int_0^{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\right.$$

$$+\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$\left.+\int_{\frac{4\bar{x}-2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\right]\frac{1}{T}\phi\,dx_2\,dx_1,$$

$$\frac{\bar{x}\sqrt{3}}{3}\le s\le \bar{x},$$

$$=32s\left[\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}\right.$$

$$+\int_{0}^{2\bar{x}-[2s^2-2\bar{x}^2]^{\frac{1}{2}}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{0}^{\frac{4\bar{x}-x_1-[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}$$

$$+\int_{2\bar{x}+[2s^2-2\bar{x}^2]^{\frac{1}{2}}}^{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}\int_{\frac{4\bar{x}-x_1+[8s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1]^{\frac{1}{2}}}{2}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}$$

$$\left.+\int_{\frac{4\bar{x}+2[6s^2-2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x}+s\sqrt{3}}\int_{\frac{4\bar{x}-x_1-2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[6s^2-2(x_1-\bar{x})^2]^{\frac{1}{2}}}{3}}\right]\frac{1}{T}\phi\,dx_2\,dx_1,$$

$$\bar{x}\le s\le \bar{x}\sqrt{3}$$

By similar reasoning, the writer has determined $F(\bar{x}, s)$ for $n = 4$ and $f(x)$ defined on the interval $(0, a)$. The results, however, are quite lengthy and formal and will not be presented here.

THE LIMITS OF A MEASURE OF SKEWNESS

By Harold Hotelling and Leonard M. Solomons, *Columbia University*

The measure of skewness

$$s = \frac{\text{mean -- median}}{\text{standard deviation}}$$

is sometimes recommended because of its simplicity. Obviously neither this nor any other statistic can be of much value until something at least is known of its distribution in samples from populations of some plausible form. For populations near the normal form the inefficiency of the median as a statistic of location suggests that the standard error of s may be considerably greater than that of μ_3/σ^3. We know of no investigation of the sampling distribution of s. Apparently even the range is unknown. The object of the present note is to show that s necessarily lies between -1 and 1.

The proof consists of three successive transformations of the sample, each increasing s, which nevertheless in the end remains less than unity.

1. Without loss of generality let us suppose that the median is zero and that the mean $\bar{x}$ is positive. Taking

$$\sigma^2 = \frac{\Sigma(x-\bar{X})^2}{n} = \frac{\Sigma x^2}{n} - \bar{x}^2,$$

n being the number of observations, which we suppose odd, we have

$$s = \bar{x}/\sigma.$$

If a negative observation $-a$ be replaced by zero, the mean is increased by a/n. In the second of the expressions above for σ^2, the mean of the squares is diminished by a^2/n, while on account of the change in the mean, a further subtraction is made. Thus σ diminishes. Hence s increases if we alter the distribution by replacing all the negative observations by zero. The median remains unchanged at zero.

2. Let us further transform this altered distribution by replacing all the positive observations by the mean of these positive quantities. The general mean is left unchanged by this transformation, but the standard deviation is diminished. For, denoting by z the deviation of a positive observation from the general mean, Σz^2 is, for a fixed value of Σz, a minimum when all the z's are equal.

3. Thus the value of s is increased when we replace all the negative observations by the median value 0 and all the positive observations by a fixed quantity, which we may take as unity. Let there be h 0's and k 1's in this distribution. Then $h+k=n$. Moreover, since the median is at 0, $h>k$. The mean is k/n. while

$$n\sigma^2 = h\cdot 0^2 + k\cdot 1^2 - k^2/n = hk/n$$

Hence $s = (k/h)^{\frac{1}{2}}$

in this case in which s is a maximum. Since as just remarked, $h>k$, this is always less than unity, approaching unity when the observations are divided as nearly as is possible equally between the two values.

To go further in a study of the sampling distribution of s is possible only on the basis of special assumptions. The same is true of the somewhat more familiar but less definite measure of skewness,

$$\frac{\text{mean} - \text{mode}}{\text{standard deviation}}.$$

It is clear that there is no limit of the range of this last quantity.

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THE THEORY OF PROBABILITY FROM THE POINT OF VIEW OF ADMISSIBLE NUMBERS

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I. INTRODUCTION

The definition of the word probability has never been agreed upon. Before we decide on a definition, let us first consider what use we hope to make of the theory of probability. It is reasonable to demand of this theory that we shall be able to apply it, and that, by means of it, we shall be able to make predictions.

If we say that the probability is .9 that a given event will occur under certain circumstances, then are we making some prediction about the success (i.e. occurrence) of the event? Let us suppose that the circumstances are presented. We may observe that the event succeeds or we may observe that it fails. Whichever the case may be, the result of the experiment cannot be interpreted in terms of the number, .9. This is always the case. We can never interpret the result of a single trial of an event in terms of the probability of that event.

Next let us assume that n trials are made of an event whose probability is .9, and that, as a result of this experiment, r successes and $n-r$ failures are obtained. If n is large, we should expect the ratio, r/n, to be approximately .9, that is, approximately nine-tenths of the trials to be successful. We shall call the number, r/n, the success ratio.

We have not even now obtained a satisfactory interpretation for the number, .9. We have not specified any limit to the discrepancy between the numbers, r/n and .9, and we have not specified the magnitude of n. Thus, if r/n differs from .9 by a small amount, it also differs from .899 by a small amount. Are we to be satisfied with the statement that the event in question has a multiplicity of probabilities including the numbers, .9 and .899?

We can make the above statement more exact as follows:

Given any positive number, $\mathcal{E}$, we can find a number, n, such that the discrepancy between r/n and .9 is less than $\mathcal{E}$. After the number, $\mathcal{E}$, has been chosen, it is at least conceivable that a sufficient number of trials can be made so that r/n will differ from .9 by less than $\mathcal{E}$. If we make this interpretation of the probability, .9, and if we wish to make the statement that .9 is *the* probability of the event, then we are assuming that .9 is the only number that has this property. We are therefore assuming that the ratio, r/n, approaches .9 as n becomes infinite.

So far as I know, no one has ever given an alternative concept of probability which is capable of being interpreted in terms of the result either of a single trial or of a sequence of trials. Unless and until such a concept is given, we are compelled to assume that probability is the limit of the success ratio, if we wish to include an emperical interpretation. Since this paper is being presented to a group of statisticians, I think it will be safe to assume that we are agreed that probability is concerned with the results of trials of events.

It may be that we arrived at the probability, .9, by means of the following reasoning. There are 9 possible ways in which the event can succeed and 1 in which it can fail. All 10 possibilities are equally likely and mutually exclusive.

When we make a trial of the event, one and only one of the possibilities succeeds. The words, equally likely, have no interpretation in terms of the result of a single trial. The reader will have little difficulty in continuing the analysis of these words in a manner similar to that of the concept of probability. In fact the concept, being equally likely, is identical with the concept, having the same probability. We shall, therefore, reject the concept of equal likelihood as a basis for a definition of probability.

There is one other objection to this method of finding the probability of an event. Namely, there is good reason to believe that it never gives the correct result. In making this statement we are assuming, of course, that probability is defined as the limit of the success ratio. In order that the 10 possibilities may be equally likely, it is necessary that there be perfect symmetry between these possibilities. We cannot, therefore, have any mark to distinguish the one unfavorable possibility from the other nine favorable possibilities. Experiment indicates that such distinguishing marks are sufficient to make noticeable differences in the probabilities. For example, the dots on the faces of a die cause differences in the frequencies with which the respective faces turn up.

In spite of these objections, the above method of finding the probability of an event, gives very good approximations in most of the cases where it is applied. There is no method which gives exact values for probabilities. It seems wise not to reject this method, but rather to discard any illusions which we may have concerning the exactness of its results.

We have seen that we must assume the probability of an event to be the limit of the success ratio, if we are agreed that probability is concerned with the results of trials. Let us express this assumption in terms of the Cauchy criterion for the existence of a limit.

Given a positive number, $\mathcal{E}$, there exists a number, N, such that $|r/n - r'/n'| < \mathcal{E}$ whenever $n \geqq N$ and $n' \geqq N$, where r is the number of successes in n trials and r' is the number of successes in n' trials. Physical experiment seems to indicate that this condition is satisfied. Furthermore, if we reject this assumption we deny the possibility of experimental verification of probabilities. On the other hand, it can be proved that the number, N, can never be known. This situation is unsatisfactory for a mathematical theory.

To avoid this difficulty we shall construct an imaginary idealized universe in much the same manner as is done in the case of geometry. This universe will contain sequences of successes and failures which are formed in accordance with mathematical laws. These sequences will satisfy the fundamental assumptions of probability and hence will be infinite. We make the assumption that the physical universe is an approximation to this idealized universe.

II. THE ALGEBRA OF EVENTS

We shall show how the elements of the theory of probability can be treated from the point of view which we have described. Consider first the following physical example. A coin is flipped ten times and the event in question is the occurrence of a head. The following is a record of the successes and failures,

$$1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0$$

where the 1's stand for successes and the 0's for failures. The ratio, $4/10$, of the number of successes to the number of trials, is obtained by adding all of the ten numbers and dividing by ten. If we had made a much larger number of trials of the event, we should expect that the corresponding success ratio would have been much closer to the probability, one-half.

The above sequence of 1's and 0's can be interpreted as a number written in the binary scale. Let us write

$$.110,\ 100,\ 010,\ 0$$

This number has the value. $1/2 + 1/4 + 0/8 + 1/16 + 0/32 + 0/64 + 0/128 + 1/256 + 0/512 + 0/1024 = 209/256$. We should not, however, think of this number as ending with the tenth digit. In fact we could compute as many more of the digits as we desired by continuing the experiment. The computation of the values of these numbers will not be important for our purposes. The above computation was inserted merely to aid in the understanding of the notation which we shall describe.

We shall now consider the construction of our idealized universe. The sequence of successes and failures of a given imaginary event can be represented by a number, $x = .x^{(1)}x^{(2)}x^{(3)}\dots x^{(k)}\dots$, written in the binary scale, the kth digit, $x^{(k)}$, of x being 1 or 0 according as the event succeeds or fails on the kth trial. We shall denote the success ratio for the first n trials of this event by $p_n(x)$.

Then

$$(1) \qquad p_n(x) = \sum_{k=1}^{n} x^{(k)}/n$$

We shall denote the probability of the event, x, by $p(x)$ and we shall define $p(x)$ by means of the equation

$$p(x) = \lim_{n \to \infty} p_n(x). \tag{2}$$

We are, of course, assuming that this limit exists.

Most of the important questions in the theory of probability involve relations between different events. We shall therefore construct an algebra which is especially adapted to the discussion of related events. If $x = .x^{(1)}x^{(2)}x^{(3)}\cdots$ and $y = .y^{(1)}y^{(2)}y^{(3)}\cdots$ are any two events, then the event, x and y, will be denoted by $x \cdot y$. We have the equation,

$$x \cdot y = .(x^{(1)} \cdot y^{(1)}), (x^{(2)} \cdot y^{(2)}), (x^{(3)} \cdot y^{(3)}) \cdots \tag{3}$$

The first digit of $x \cdot y$ is 1 if and only if the first digits of x and y are both 1. That is, the event, $x \cdot y$, succeeds on the first trial if and only if x and y both succeed on the first trial. Similarly for the second and third trials etc. The expressions inside the parentheses are understood to be ordinary algebraic products. The expression, $x \cdot y$, is a symbolic product.

The event, x or y or both, is denoted by $x \vee y$. We have the equation

$$x \vee y = (x^{(1)} + y^{(1)} - x^{(1)} \cdot y^{(1)}), (x^{(2)} + y^{(2)} - x^{(2)} y^{(2)}), \cdots \tag{4}$$

It will be observed that the first digit, $(x^{(1)} + y^{(1)} - x^{(1)} \cdot y^{(1)})$ of $x \vee y$ is 1 if $x^{(1)} = y^{(1)} = 1$ or if $x^{(1)} = 1$, $y^{(1)} = 0$ or if $x^{(1)} = 0$, $y^{(1)} = 1$, but that this digit is 0 if $x^{(1)} = y^{(1)} = 0$. Thus the event, $x \vee y$, succeeds on the first trial if x succeeds on its first trial or y succeeds on its first trial or both x and y succeed on their first trials. Similarly for the second and third trials etc.

We shall use the symbol, $\sim x$, to denote the event, not x. It is easily seen that $\sim x$ is given by the equation,

$$(5) \qquad \sim x = .\,(1-x^{(1)}),\ (1-x^{(2)}),\ (1-x^{(3)}),\ \dots\ .$$

Let us denote the event, y if x, by $y \subset x$.* Before attempting to give a formula for $y \subset x$ let us first consider the expression, $m \cdot p_m(x)$. This expression is equal to the number of successes of the event, x, in its first m trials. Thus if m_n is the number of the trial on which the nth success of x occurs, then

$$(6) \qquad m_n \cdot p_{m_n}(x) = n.$$

We can write

$$(7) \qquad y \subset x = .\, y^{(m_1)} y^{(m_2)} y^{(m_3)} \dots\ .$$

Thus we consider those trials of y for which the event, x, occurs. In other words we consider a given trial of y if (and only if) x occurs on that trial. Hence equation (7) gives us the correct expression for the event, y if x.

[*The operators, $\cdot$, $\vee$ and $\sim$ are also used in symbolic logic with similar interpretations. See Whitehead and Russell, *Principia Mathematica,* vol. 1. The symbol, $\subset$, is an inverted implication sign. The expression, $y \subset x$, could be read, y is implied by x, or, y if x. For the benefit of those who are familiar with *Principia Mathematica,* it may be added that the symbols, x, y, etc. are propositional functions rather than propositions. Each x is associated with a sequence of events, and each is a propositional function of the form, the kth event will succeed, k being a free variable. The probability is a property of the set of propositions rather than of any given proposition. Thus we should speak of the probability of a propositional function rather than of the probability of a proposition.]

PROBLEMS

In problems, 1 to 3, assume that x and y have the following values

$$x = .110,100,010,011,101,000,10\cdots$$
$$y = .110,111,011,000,100,010,11\cdots$$

1. Compute $\rho_{15}(x)$ and $\rho_{20}(y)$.
2. Compute the first 20 digits of (a) $x\cdot y$, (b) $x\vee y$, (c) $\sim x$, (d) $y\cdot\sim x$.
3. Compute as many digits as possible of $y\subset x$ and $x\subset y$.
4. Prove the following identities:

   (a) $x\cdot y = y\cdot x$ (g) $\sim\sim x = x$

   (b) $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ (h) $\sim(x.y) = \sim x\vee\sim y$

   (c) $x\vee y = y\vee x$ (i) $\sim(x\vee y) = \sim x\cdot\sim y$

   (d) $x\vee(y\vee z) = (x\vee y)\vee z$ (j) $x\vee\sim x = 1$

   (e) $x\cdot(y\vee z) = (x\ y)\vee(x\cdot z)$ (k) $(x\ y)\vee(x\cdot\sim y) = x$

   (f) $x\vee(y\cdot z) = (x\vee y)\cdot(x\vee z)$

5. Prove that $\rho_n(x\vee y) = \rho_n(x)+\rho_n(y)-\rho_n(x\cdot y)$
6. Prove that $\rho(x\vee y) = \rho(x)+\rho(y)-\rho(x\cdot y)$
7. Prove that $\rho(\sim x) = 1-\rho(x)$
8. Prove that $\rho[y\cdot\sim x] = \rho(y)-\rho(x\cdot y)$
9. Prove that if $x\cdot\sim(y\vee z\vee w) = 0$ then $x = (x\cdot y)\vee(x\cdot z)\vee(x\ w)$

III. THE COMPUTATION OF PROBABILITIES

We shall say that two events, x and y, are mutually exclusive provided x fails whenever y occurs and y fails whenever x occurs. It is easily seen that x and y are mutually exclusive if and only if $x\cdot y=0$. It follows from problem (6) that

$$(8)\qquad p(x\vee y)=p(x)+p(y), \text{ if } x\cdot y=0.$$

If we have three events, x, y, and z, which are mutually exclusive, then $x\cdot y=y\cdot z=z\cdot x=0$. Hence

$$p(x\vee y\vee z)=p(x\vee y)+p(z)=p(x)+p(y)+p(z).$$

We have the following theorem.

Theorem 1. If the events, $x_1, x_2, x_3, \ldots x_n$, are mutually exclusive then

$$p(x_1\vee x_2\vee\cdots\vee x_n)=p(x_1)+p(x_2)+p(x_3)+\cdots+p(x_n).$$

Suppose we have a set of events, $x_1, x_2, \cdots x_n$, such that at least one of the events must occur. Then $x_1\vee x_2\vee x_3\vee\cdots\vee x_n=1$. Suppose further that these events are mutually exclusive and that their probabilities are equal. Then $p(x_1)+p(x_2)+\cdots+p(x_n)=1$ and therefore $p(x_1)=p(x_2)=\cdots=p(x_n)=1/n$. This principle is very useful in the computation of probabilities.

Example 1. From a pack of 52 cards 1 card is drawn. What is the probability that this card is the ace of spades? It is reasonable to assume that the probability of drawing any one of the 52 cards, is the same as that of drawing any other card. Thus we have 52 events which have the same probabilities. Moreover these events are mutually exclusive and it is a certainty that at least one of the events will occur. Hence the desired probability is $1/52$.

Example 2. From a pack of 52 cards, 13 cards are drawn. What is the probability that these cards are all spades? We assume that any combination of 13 cards has the same probability as any other combination of 13 cards. Since there are ${}_{52}C_{13}$ such combinations, the probability is $1/{}_{52}C_{13}=1/635,013,559,600$.

We shall now compute the probability of the event, $y\subset x$.

We have the equations,

$$(9)\quad p_n(y\subset x)=\frac{\sum\limits_{i=1}^{n} y^{m_i}}{n}=\frac{\sum\limits_{k=1}^{m_n}\frac{x^k y^k}{m_n}}{\frac{n}{m_n}}=\frac{p_{m_n}(x\cdot y)}{p_{m_n}(x)}$$

where $m_n\cdot p_{m_n}(x)=n.$

If we allow n to become infinite we get

$$(10)\qquad p(y\subset x)=p(x\cdot y)/p(x).$$

Multiplying both sides of equation (10) by $p(x)$ we get

$$(11)\qquad p(x)\cdot p(y\subset x)=p(y\cdot x).$$

Example 3. A pack of 52 cards is divided into 4 piles of 13 cards each. One pile contains just 1 heart and the other 3 piles contain 4 hearts each. A pile is selected at random and a card is drawn from this pile. What is the probability that the pile selected will be the one containing just the one heart and that the card selected from this pile will be the heart? Let y represent the drawing of a heart and x represent the drawing of the pile containing just one heart. Then $p(x)=1/4$ and $p(y\subset x)=1/13.$ Hence $p(y\cdot x)=p(x)\cdot p(y\subset x)=1/52.$ This is the desired probability.

We shall say that an event, y, is independent on an event, x, provided the probability that y will occur is the same whether x occurs or not. If we express this condition for independence in terms of our symbols we will get

$$(12)\qquad p(y\subset x)=p(y\subset \sim x).$$

Hence

$$(13)\qquad \frac{p(y\cdot x)}{p(x)}=\frac{p(y\cdot\sim x)}{p(\sim x)}=\frac{p(y)-p(y\cdot x)}{1-p(x)}$$

Therefore

$$(14)\qquad p(x\cdot y)=p(x)\cdot p(y)$$

It is a simple matter to reverse our steps and start with equation (14) and obtain equation (12). Moreover, from the symmetry of equation (14) it is easily seen that if y is independent of x then x is independent of y. We have now proved the following theorem.

Theorem 2. A necessary and sufficient condition that two events, x and y, be independent, is that $p(x\cdot y)=p(x)\cdot p(y)$.

Example 4. A coin and a die are thrown together. What is the probability that the coin will turn up a head and the die will turn up a 3? Let x represent the occurrence of a head and y represent the occurrence of a 3. Then $p(x)=1/2$ and $p(y)=1/6$. Since the events are independent it follows that $p(x\, y)=1/12$

It should be observed that equation (11) is always true but that equation (14) can only be used when the two events are independent. The term, contingent, is used to apply to events which are not independent. If x and y are two contingent events we must use equation (11) to compute $p(x\cdot y)$.

In order that three events, x, y, z, may be independent, it is necessary and sufficient that $p(x\cdot y)=p(x)p(y)$,

$$p(y\cdot z)=p(y)p(z),\ p(z\cdot x)=p(z)p(x),\ p(x\cdot y\cdot z)$$
$$=p(x)p(yz)=p(y)p(z\, x)=p(z)p(x\cdot y).$$

This definition is easily generalized to the case of n events.

It is generally assumed that the trials of an event are independent. What does this assumption mean? Suppose, for example, that we wish to say that the first trial of an event is independent of the second. The first trial constitutes an event, x_1, and the second trial constitutes an event, x_2, but we have only defined one trial of x_1 and one trial of x_2. Independence is defined in terms of probabilities, and probabilities can be given meaning only in terms of sequences of trials.

We can get around the difficulty in the following manner. Suppose we wish to consider the independence of n trials of an event, x. We will consider n events, $x_1, x_2, x_3, \ldots x_n$. The first trial of x_1 will be the first trial of x, the first trial of x_2 will be the second trial of x, the first trial of x_3 will be the third trial of x, etc. The 2nd trial of x_1 will be the $(n+1)st$ trial of x, the 2nd trial of x_2 will be the $(n+2)nd$ trial of x, etc. In general, the digits of the number, x_r, are selected from the digits of the number, x. The digits selected are, the rth, the $(r+n)th$, the $(r+2n)th$, $(r+3n)th$, etc. That is

$$(15) \qquad x_r = .\,x^{(r)}x^{(r+n)}x^{(r+2n)}x^{(r+3n)} \ldots$$

We can now speak of the independence of the numbers, x_1, x_2, $\ldots x_n$.

It will be observed that

$$(16) \qquad \frac{2^{-r}}{1-2^{-n}} = 2^{-r}+2^{-r-n}+2^{-r-2n}+2^{-r-3n}+\ldots$$

and hence we can write

$$(17) \qquad x_r = x\, c\, \frac{2^{-r}}{1-2^{-n}}$$

We shall abbreviate this notation still further and write

$$(18) \qquad (r/n)x = x\, c\, \frac{2^{-r}}{1-2^{-n}} .$$

It is natural to assume that $p[(r/n)x] = p(x)$ for every pair of numbers, r and n, such that $0 < r \leqq n$. If we assume this, and if we assume that the numbers, $(1/n)x, (2/n)x, (3/n)x \ldots (n/n)x$, are independent, then x must satisfy the following equations.

$$(19) \qquad p[(r_1/n)x\cdot(r_2/n)x\cdots(r_k/n)x] = [p(x)]^k$$

for every n and for every set of integers, $r_1, r_2, r_3, \ldots r_k$, such that $0 < r_1 < r_2 < \cdots < r_k \leqq n$.

Any number, x, which satisfies equations (19) is called an admissible number. It can be proved that there exist admissible numbers.* It is clear that an admissible number, x, characterizes the behavior which we should expect from a sequence of trials of an event with probability, $p(x)$.

[*See the author's article, *Admissible numbers in the theory of probability*, American Journal of Mathematics, Vol. L., No. 4, Oct. 1929].

Example 5. An event, x, has the probability, $p(x)$. What is the probability of obtaining precisely two successes in three trials of the event? It is required to find

$$p\{[(1/3)x\cdot(2/3)x \ \sim(3/3)x]\vee[(2/3)x\cdot(3/3)x \sim(1/3)x]$$
$$\vee[(3/3)x \ (1/3)x\cdot\sim(2/3)x]\}.$$

Each of the square brackets contains three independent numbers. Thus for each square bracket we have the probability, $[p(x)]^2 p(\sim x)$. The square brackets themselves constitute three mutually exclusive events. Hence the desired probability is $3[p(x)]^2 p(\sim x)$.

Let us find the probability of r successes and $n-r$ failures in n trials of an event. Let $p(x)=p$ and $p(\sim x)=q$. The probability that a given set of r trials will all be successful, is p^r, and the probability that the remaining $n-r$ trials will all be failures, is q^{n-r}. The r successful trials can be chosen in ${}_nC_r$ ways. Since all of these ways are mutually exclusive, the desired probability is ${}_nC_r\, p^r q^{n-r}$.

Consider the following problem. Let $x_1, x_2, \ldots x_n$ be a set of mutually exclusive events whose probabilities are known. We shall call these events causes. Let y be an event which can occur only as a result of one of the causes. The probabilities of y if x_1, y if x_2, etc. are also known. An experiment is performed and it is observed that y occurs. What is the probability that this occurrence is a result of kth cause? The answer to this question is given by the following theorem.

Theorem 3. If $x_1, x_2, x_3, \ldots x_n$ is a set of mutually exclusive events, and if y is such that $y\cdot\sim(x_1\vee x_2\vee\cdots\vee x_n)=0$, then

$$p(x_k\subset y)=\frac{p(x_k)\cdot p(y\subset x_k)}{\sum_{i=1}^{n} p(x_i)\cdot p(y\subset x_i)}.$$

Since $y \sim(x_1\vee x_2\vee\cdots\vee x_n)=0$ it follows that

$$y=(y\cdot x_1)\vee(y\ x_2)\vee\cdots\vee(y\cdot x_n).$$

Hence $p(y)=p(y\cdot x_1)+p(y\ x_2)+\cdots+p(y\ x_n)$.

Therefore

$$p(y) = p(x_1)\,p(y \subset x_1) + p(x_2)\,p(y \subset x_2) + \cdots + p(x_n)\,p(y \subset x_n).$$

To complete the proof of the theorem it is only necessary to substitute this value of $p(y)$ in the equation, $p(x_k \subset y) = p(x_k \cdot y)/p(y)$, and then substitute $p(x_k)\,p(y \subset x_k)$ for $p(x_k\ y)$.

Theorem 3 is known as Bayes' principle The probabilities, $p(x_1), p(x_2), \ldots p(x_n)$, are called *a priori* probabilities, whereas the probabilities, $p(x_1 \subset y), p(x_2 \subset y), \ldots p(x_n \subset y)$, are called *a postiori*.

Example 6. There are four urns, U_0, U_1, U_2, U_3. The urn, U_0, contains three black balls, U_1 contains one white ball and two black balls, U_2 contains two white and one black, and U_3 contains three white balls. An urn is selected at random and a ball is drawn from it and found to be white. What is the probability that the ball came from U_2? Let x_0, x_1, x_2, x_3 represent respectively the drawing of U_0, U_1, U_2, U_3, and let y represent the drawing of a white ball from the urn selected. Then

$$p(x_0) = p(x_1) = p(x_2) = p(x_3) = 1/4,$$

and $p(y \subset x_0) = 0$, $p(y \subset x_1) = 1/3$, $p(y \subset x_2) = 2/3$, $p(y \subset x_3) = 3/3$.

Hence $$p(x_2 \subset y) = \frac{\frac{1}{4}\cdot\frac{2}{3}}{\frac{1}{4}\,\frac{1}{3} + \frac{1}{4}\,\frac{2}{3} + \frac{1}{4}\,\frac{3}{3}} = 1/3.$$

Example 7 Two people, A and B, make the same statement independently. Let this event be denoted by y. Let x denote the event that the statement is true. Then y can be the result of two causes, $x_1 = x$ and $x_2 = \sim x$. It is given that the probabilities of A and B speaking the truth, are respectively a and b. What is the *a postiori* probability that the statement is true? We know that

$p(y \subset x_1) = ab$ and $p(y \subset x_2) = (1-a)(1-b)$. Hence

$$p(x \subset y) = \frac{a . b\ p(x)}{a \cdot b \cdot p(x) + (1-a)(1-b) \cdot p(\sim x)}$$

It might be added by way of warning that it is easy to state a problem of this kind, which is without meaning.

Let us consider the problem of finding the probability that an event, x, will preceed an event, y, a tie being excluded. We have the four possible situations, $x \cdot y$, $(\sim x)\cdot(\sim y)$, $x \cdot (\sim y)$, $(\sim x)\cdot y$. The first situation represents the tie, and this situation we have excluded. In the second situation neither x nor y succeeds, and this situation should also be excluded. The event, x, will preceed y provided x succeeds and y fails if either of the last two situations occurs. Hence the desired probability is

$$p[(x\cdot\sim y)\subset\{(x\cdot\sim y)\vee(y\cdot\sim x)\}] = \frac{p(x\cdot\sim y)}{p(x\cdot\sim y)+p(y\,\sim x)}\,.$$

When x and y are mutually exclusive this last expression takes the following form:

$$\frac{p(x)}{p(x)+p(y)}\,.$$

IV. CONCLUSION

The above examples illustrate how the theory of probability can be developed in terms of our idealized universe. By this method we can construct a consistent mathematical theory, and one which admits the possibility of experimental verification.

RELATIVE RESIDUALS CONSIDERED AS WEIGHTED SIMPLE RESIDUALS IN THE APPLICATION OF THE METHOD OF LEAST SQUARES

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In a recent paper the writer[1] discussed some considerations involved in fitting a curve, by the method of least squares, to data in which the magnitude of the errors of measurement was affected by the size of the dependent variable. For the special case in which the percentage errors of measurement were distributed normally, it was shown that the most probable values of the dependent variable could be calculated by minimizing the sum of the squares of residuals of the type, $V - \frac{Y}{f(X)}$, with respect to V, V being the arithmetic mean of the ratios of the observed values of the dependent variable to the corresponding calculated values and equal to unity at that minimum.

The concept of a relative residual has a certain value to the investigator as an aid in visualizing the nature of such a set of data. However, it is possible to use a different method of analysis, based on the theory of weighting, which will yield exactly the same results when applied to such a set of data and in addition possesses the advantage of being applicable to more general problems in which the relation of the errors of measurement to the values of the dependent variable is more complex.

Standard texts on the method of least squares such as that by Merriman,[2] show that if the probability of the occurrence of an error of a given magnitude varies for measurements of successive values of the dependent variable, it is necessary to weight the observation equations when fitting the curve. If the errors of measurement, made in obtaining one observed value of each of several successive values of a dependent variable, $f(X)$, are influenced by the magnitude of, $f(X)$, the probability of the occurrence of errors of the magnitudes, $x_1, x_2, x_3, \ldots \ldots, x_n$, respectively, is given by the following equations:

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[1] Hendricks, Walter A. 1931. The use of the relative residual in the application of the method of least squares. *Annals of Mathematical Statistics*, 2 (4): 458-478.

[2] Merriman, Mansfield. 1907. The method of least squares. 230 p., illus. John Wiley & Sons, New York.

$$P_1 = k_1 e^{-h_1^2 x_1^2}$$

$$P_2 = k_2 e^{-h_2^2 x_2^2}$$

$$P_3 = k_3 e^{-h_3^2 x_3^2}$$

$$P_n = k_n e^{-h_n^2 x_n^2} \quad \ldots\ldots \quad (1)$$

The probability of the occurrence of the given system of errors is given by the product:

$$P' = k' e^{-(h_1^2 x_1^2 + h_2^2 x_2^2 + h_3^2 x_3^2 + \cdots h_n^2 x_n^2)} \quad (2)$$

in which $P' = P_1 . P_2 . P_3 . \; P_n$ and $k' = k_1 . k_2 \; k_3 \cdots k_n$.

If the exponent of e in equation (2) is divided by a constant measure of precision, h^2, the equation may be written in the form:

$$P' = k' e^{-h^2(p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + \quad p_n x_n^2)} \quad (3)$$

in which $h_1^2 = p_1 h^2$, etc., and $p_1, p_2, p_3, \cdot p_n$ are the weights of the corresponding errors. P' will have its maximum value when the value of the expression, $p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + \cdots p_n x_n^2$, is a minimum.

Applying the above principles to curve fitting and substituting a residual, v, for every error, x, to distinguish the residuals from the true errors, it is evident that the constants of a fitted equation must be determined in such a manner that the value of the expression, $p_1 v_1^2 + p_2 v_2^2 + p_3 v_3^2 +$ $p_n v_n^2$, is a minimum.*

If the equation to be fitted is of the type used in the writer's previous study (loc. cit.), viz.:

$$Y = AX^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

this condition is obviously satisfied by the solution of the following equation:

$$p_1 v_1 \frac{\partial v_1}{\partial A} + p_2 v_2 \frac{\partial v_2}{\partial A} + p_3 v_3 \frac{\partial v_3}{\partial A} + \quad \cdot p_n v_n \frac{\partial v_n}{\partial A} = 0 \text{ -- } (5)$$

Let Y_i represent any observed value of the dependent variable and let AX_i^2 represent the corresponding most probable value. Then it is evident that:

$$v_i = AX_i^2 - Y_i \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and

$$\frac{\partial v_i}{\partial A} = x_i^2 \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

All that remains is to find the weight, p_i .

Equations (2) and (3) show that the weights of the errors are proportional to the respective measures of precision, h_1^2, h_2^2, h_3^2, $---h_n^2$. It follows from the well-known relation between the measures of precision and variance that any measure of precision, h_i^2, is equal to $\frac{1}{2\sigma_i^2}$ in which σ_i is the standard error of the observed value, Y_i , of the dependent variable. Therefor, any weight, p_i , is given by the equation:

$$p_i = \frac{1}{2h^2\sigma_i^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

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*The above development follows that given by Merriman with a slight change in notation.

in which h^2 is the constant measure of precision in equation (3).

If a set of data were obtained by making one measurement of each of several successive values of the dependent variable, AX^2, and the resulting percentage errors of measurement were distributed normally, it follows that the coefficient of variation of a number of replicate measurements made at any value of AX^2 would be equal to that obtained for every other value of AX^2. In other words, the standard error of every measurement would be directly proportional to the value of AX^2 measured.

Since, by equation (8), the weight of any error of measurement is inversely proportional to the square of its standard error, it follows from the above discussion that this weight must also be inversely proportional to the square of the value of AX^2 measured. Combining all factors of proportionality into a composite constant, C, the relation between any weight, p_i, and the corresponding value of the dependent variable, AX_i^2, may be expressed by the equation:

$$p_i = \frac{C}{A^2X_i^4} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

If the substitutions suggested by equations (6), (7), and (9) are made in equation (5), this equation may be written:

$$\frac{C(AX_1^2-Y_1)X_1^2}{A^2X^4} + \frac{C(AX_2^2-Y_2)X_2^2}{A^2X^4} + \frac{C(AX_3^2-Y_3)X_3^2}{A^2X^4} + \cdots$$

$$\frac{C(AX_n^2-Y_n)X_n^2}{A^2X^4} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

Since the constant, C, is common to every term in equation (10), it may be removed by division. The equation may then be reduced to the form:

$$\Sigma \frac{(AX^2-Y)X^2}{A^2X^4} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

If there are n observation equations, equation (11) may be written in the form.

$$\frac{n}{A} - \frac{1}{A^2} \Sigma \frac{Y}{X^2} = 0 \qquad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

from which the most probable value of A may be readily calculated.

$$nA - \Sigma \frac{Y}{X^2} = 0$$

or

$$A = \frac{1}{n} \Sigma \frac{Y}{X^2} \qquad . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

Equation (13) is identical with the equation obtained by minimizing residuals of the type, $V - \frac{Y}{AX^2}$, reported in the writer's earlier study (loc. cit.). The development given in the present paper is perhaps the better from the purely mathematical point of view since it involves nothing more than a systematic weighting of the observation equations. It can be applied to any problem in curve fitting if the standard error of each observed value of the dependent variable is known or can be deduced from a priori considerations .For example, it often happens that the means of replicated measurements, rather than the individual measurements themselves, are used in fitting the curve. In such cases the standard error of each mean may easily be calculated. The reciprocals of the squares of these standard errors will then be the required weights of the observation equations

However, if the standard errors are proportional to the values of the dependent variable, it may be desirable to retain the concept of a relative residual. The significance of a percentage error of measurement probably can be appreciated by many investigators in various fields of research, particularly those whose contact with mathematics is more or less incidental, to whom a system of weighting would seem somewhat artificial and arbitrary.

In either event, the necessary computations are identical. The precise procedure described in the present paper, like that developed in the writer's previous study (loc. cit.), cannot be applied when the equation which is to be fitted contains more than one undetermined constant. However, in actual practice it is usually sufficiently accurate to substitute the square of the observed value of the dependent variable for that of the corresponding most probable value in equation (9). If this is done, the method can be applied to any equation which can be fitted by the method of least squares. Using this substitution is equivalent to expressing the errors of measurement as fractions of the observed values of the dependent variable when the standard error of each measurement is proportional to the quantity measured.

MOMENTS AND DISTRIBUTIONS OF ESTIMATES OF POPULATION PARAMETERS FROM FRAGMENTARY SAMPLES*

S. S. WILKS**

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**National Research Fellow in Mathematics.

I. Introduction.

It frequently happens that all of the individuals of a sample of statistical data from a multivariate population are not observed or classified with respect to all of the variates. If a sample be represented in matrix form by allowing the rows to represent the individuals and the columns to represent the variates, then the matrix of the type of sample with which we are concerned is incomplete in that some of the elements are not present. As an example of a fragmentary sample of this nature, we may consider a series of measurements taken from certain parts of a group of human skeletons from some archeological find, in which some of the parts under consideration are missing from some of the skeletons. Again, we find such a class of samples in the social sciences and government statistics arising from incompletely answered questionnaires.

In dealing with fragmentary samples, it is important to have at hand techniques which will enable the investigator to extract as much information as possible from the data. This is especially true if the data are unique or expensive. An important problem in this connection is that of estimating the population parameters from the sample.

In this paper it is the purpose of the author to investigate incomplete samples from a normal bivariate population. To be more specific, samples are considered from a normal bivariate population of x and y, in which s of the items are observed with respect to x and y, m with respect to x only and n with respect to y only. In the first part of the paper we shall consider various sets of simultaneous maximum likelihood estimates of the population parameters and the limiting forms of their sampling variances and covariances in large samples. In the second part we shall consider other less efficient, but simpler systems of estimates.

II. Simultaneous Estimation by the Method of Maximum Likelihood.

Let a sample ω of N individuals be drawn from the population of the two variates x and y whose distribution is given by

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}\,e^{-\frac{1}{2(1-r^2)}\left[\frac{(x-a)^2}{\sigma_x^2}+\frac{(y-b)^2}{\sigma_y^2}-\frac{2r(x-a)(y-b)}{\sigma_x\sigma_y}\right]},$$

where a and b are the means, σ_x and σ_y the standard deviations and r the correlation of x and y in the population. Let ω_{xy} be the set of s individuals of this sample observed with respect to x and y, ω_x the set of m items observed with respect to x only and ω_y the remaining n items observed with respect to y only. To avoid trivial results we shall assume that s is not zero. Furthermore, we shall let ξ and η be the variances and ζ the convariance[1], $\bar{x}$ and $\bar{y}$ the means of x and y in ω_{xy}. The variance and mean of x in ω_x will be denoted by u and $\bar{x}_1$ respectively, and similarly, the variance and mean of y in ω_y will be v and $\bar{y}_1$. The joint distribution of $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$, u, v, ξ, η and ζ can be written from several well known independent distributions as

(1)
$$F=K(\sigma_x)^{-s-m}(\sigma_y)^{-s-n}(1-r^2)^{-\frac{s}{2}}\,e^{-\frac{s}{2(1-r^2)}\left[\frac{\xi+(\bar{x}-a)^2}{\sigma_x^2}+\frac{\eta+(\bar{y}-b)}{\sigma_y^2}-2r\frac{\zeta+(\bar{x}-a)(\bar{y}-b)}{\sigma_x\sigma_y}\right]}\,e^{-\frac{m}{2\sigma_x^2}\left[u+(\bar{x}_1-a)^2\right]-\frac{n}{2\sigma_y^2}\left[v+(\bar{y}_1-b)^2\right]}$$
$$u^{\frac{m-3}{2}}v^{\frac{n-3}{2}}(\xi\eta-\zeta^2)^{\frac{s-4}{2}},$$

where

$$K=\frac{(m)^{\frac{m}{2}}(n)^{\frac{n}{2}}s^{s}\,2^{-\frac{m+n+2s}{2}}}{(\pi)^{\frac{1}{2}}\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s-2}{2}\right)}.$$

---

[1] If, for two sets of variates, $t_{11}, t_{12}, \cdots t_{1n}$; $t_{21}, t_{22}, \cdots t_{2n}$ we let $\bar{t}_j=\frac{1}{n}\sum_{i=1}^{n}t_{ji}$ and $v_{jk}=\frac{1}{n}\sum_{i=1}^{n}(t_{ji}-\bar{t}_j)(t_{ki}-\bar{t}_k)$, $(j, k=1,2)$. then $\bar{t}_1$ and $\bar{t}_2$ are the means, v_{11} and v_{22} are the variances and v_{12} is covariance of the two sets of t's.

The likelihood of ω when ω is specified in terms of the foregoing statistics is given by (1). We shall use this expression of the likelihood to obtain approximations for the maximum likelihood estimates of the population parameters σ_x, σ_y, r, a and b. Following Fisher[2], we shall take the logarithm of (1) and denote it by L. For convenience, we shall, once for all, set up the following set of first derivatives,

$$(e_1)\ \frac{\partial L}{\partial a} = \frac{s}{\sigma_x}\left[\frac{(\bar{x}-a)}{\sigma_x(1-r^2)} + \frac{\alpha(\bar{x}_1-a)}{\sigma_x} - \frac{r(\bar{y}-b)}{\sigma_y(1-r^2)}\right]$$

$$(e_2)\ \frac{\partial L}{\partial b} = \frac{s}{\sigma_y}\left[\frac{(\bar{y}-b)}{\sigma_y(1-r^2)} + \frac{\beta(\bar{y}_1-b}{\sigma_y} - \frac{r(\bar{x}-a)}{\sigma_x(1-r^2)}\right]$$

$$(2)\ (e_3)\ \frac{\partial L}{\partial \sigma_x} = \frac{s}{\sigma_x}\left[-(1+\alpha) + \frac{1}{1-r^2}\left(\frac{\bar{\xi}}{\sigma_x^2} - \frac{r\bar{\zeta}}{\sigma_x\sigma_y}\right) + \frac{\alpha\bar{u}}{\sigma_x^2}\right]$$

$$(e_4)\ \frac{\partial L}{\partial \sigma_y} = \frac{s}{\sigma_y}\left[-(1+\beta) + \frac{1}{1-r^2}\left(\frac{\bar{\eta}}{\sigma_y^2} - \frac{r\bar{\zeta}}{\sigma_x\sigma_y}\right) + \frac{\beta\bar{v}}{\sigma_y^2}\right]$$

$$(e_5)\ \frac{\partial L}{\partial r} = \frac{s}{1-r^2}\left[r + \frac{\bar{\zeta}}{\sigma_x\sigma_y} - \frac{r}{1-r^2}\left(\frac{\bar{\xi}}{\sigma_x^2} + \frac{\bar{\eta}}{\sigma_y^2} - \frac{2r\bar{\zeta}}{\sigma_x\sigma_y}\right)\right],$$

where $\bar{\xi} = \xi + (\bar{x}-a)^2$, $\bar{\eta} = \eta + (\bar{y}-b)^2$, $\bar{u} = u + (\bar{x}_1-a)^2$

$$\bar{v} = v + (\bar{y}_1-b)^2,\quad \bar{\zeta} = \zeta + (\bar{x}-a)(\bar{y}-b),\quad \alpha = \frac{m}{s},\ \beta = \frac{\eta}{s}.$$

In order to consider the limiting form of the sampling variances and covariances of the maximum likelihood estimates, we shall need the matrix of mathematical expectations of the second derivatives of L with respect to the five population parameters. This matrix of expected values turns out to be,

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[2] R. A. Fisher, The Mathematical foundations of theoretical statistics. Philosophical Transactions of the Royal Society of London, Vol. 222 (1922), pp. 309-368.

(3)

$$\begin{array}{c|c|c|c|c|c|}
 & \frac{\partial L}{\partial z_1} & \frac{\partial L}{\partial z_2} & \frac{\partial L}{\partial z_3} & \frac{\partial L}{\partial z_4} & \frac{\partial L}{\partial z_5} \\
\hline
\frac{\partial L}{\partial z_1} & \frac{s}{\sigma_x^2}\left(\frac{1}{1-r^2}+\alpha\right) & -\frac{sr}{\sigma_x\sigma_y(1-r^2)} & 0 & 0 & 0 \\
\hline
\frac{\partial L}{\partial z_2} & -\frac{sr}{\sigma_x\sigma_y(1-r^2)} & \frac{s}{\sigma_y^2}\left(\frac{1}{1-r^2}+\beta\right) & 0 & 0 & 0 \\
\hline
\frac{\partial L}{\partial z_3} & 0 & 0 & \frac{s[2\alpha(1-r^2)+(2-r^2)]}{\sigma_x^2(1-r^2)} & -\frac{sr^2}{\sigma_x\sigma_y(1-r^2)} & -\frac{sr}{\sigma_x(1-r^2)} \\
\hline
\frac{\partial L}{\partial z_4} & 0 & 0 & -\frac{sr^2}{\sigma_x\sigma_y(1-r^2)} & \frac{s[2\beta(1-r^2)+(2-r^2)]}{\sigma_y^2(1-r^2)} & -\frac{sr}{\sigma_y(1-r^2)} \\
\hline
\frac{\partial L}{\partial z_5} & 0 & 0 & \frac{sr}{\sigma_x(1-r^2)} & -\frac{sr}{\sigma_y(1-r^2)} & \frac{s(1+r^2)}{(1-r^2)^2} \\
\hline
\end{array}$$

where the entry in the i-th row and j-th column is $-E\left(\frac{\partial^2 L}{\partial z_i\,\partial z_j}\right)$ where z_i is identical with a, b, σ_x, σ_y, and r as i takes the values 1, 2, . . 5 respectively. Again, α and β denote the ratios $\frac{m}{s}$ and $\frac{n}{s}$ which we shall consider constant as $s \to \infty$.

The maximum likelihood estimates of any number of the five population parameters for given values of the remaining parameters are to be found by setting the corresponding first derivatives in (2) equal to zero and solving the resulting equations simultaneously. In most of the cases of practical interest, the solutions must be reached by approximation. In this paper we shall consider the following cases:

1. Estimation of a and b for given estimates of σ_x, σ_y

and r.

2. Estimation of σ_x and σ_y for given estimates of a, b and r.
3. Estimation of σ_x, σ_y and r for given values of a and b.

Before proceeding with the maximum likelihood estimates of these parameters we shall consider the notion of the efficiency of a set of statistics designed to estimate a set of population parameters.

1. Joint efficiency of a set of estimates.

In order to attach an economic value to a sample and its individuals, Fisher[a] has defined the reciprocal of the variance of a maximum likelihood statistic w of a sample from a univariate population as the amount of information contained in the sample relative to the population value of w. For large samples, in which the distribution of w tends to normality, this quantity is a constant multiple of the number of items in the sample. The amount of information contributed by each member of the sample can be found by dividing by the number in the sample.

We can extend the idea of amount of information relative to a system of population parameters contained in a sample by considering the reciprocal of the determinant of the limiting values, in large samples, of the variances and covariances of the maximum likelihood estimates of this system of parameters. This extended definition also holds for systems of parameters estimated from multivariate populations.

The reason for adopting this determinant as the extension of the idea of the amount of information relative to the set of parameters under consideration, is apparent when we note that the square root of its reciprocal enters as a multiplier in the asymptotic normal distribution of the maximum likelihood estimates of the parameters in the same way that the square root of the reciprocal of the sampling variance of the maximum likelihood estimate of a single parameter enters as a multiplier in its asymptotic normal distribution.

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[a] R. A. Fisher, Statistical methods for research workers, third edition, Oliver and Boyd (1930) pp. 266-270.

Fisher[4] has shown that for large samples, the maximum likelihood estimate of a population parameter is distributed with smaller variance than any other statistic designed to estimate the same parameter. In the case of a set of parameters, the determinant of the matrix of limiting values, in large samples, of the variances and covariances of the maximum likelihood estimates of the parameters is smaller than that for any other estimates of the same set of parameters.

To prove this, let us consider a set $\{p_i\}$, $(i=1,2,\cdots n)$ of population parameters, and let the set $\{t_i\}$ be their maximum likelihood estimates, whose sampling distribution for large samples is

$$\frac{\sqrt{H}}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}\sum_{i,j=1}^{n} h_{ij}(t_i-p_i)(t_j-p_j)}$$

where $H = |h_{ij}|$, where $h_{ij} = -E\left(\frac{\partial^2 L}{\partial p_i \partial p_j}\right)$ and L is the logarithm of the likelihood of the sample. H is the reciprocal of the matrix of variances and covariances and covariances of the t's. Let the set $\{u_i\}$ be any set of estimates of $\{p_i\}$ in which at least one u is not a maximum likelihood estimate, and let the asymptotic normal distribution of the u's be,

$$\frac{\sqrt{K}}{(2\pi)^{\frac{n}{2}}}\, e^{-\frac{1}{2}\sum_{i,j=1}^{n} k_{ij}(u_i-p_i)(u_j-p_j)}$$

where $K = |k_{ij}|$, which is the reciprocal of the determinant of the matrix of variances and covariances of $\{u_i\}$. Now, our problem is equivalent to that of showing that $H > K$. Suppose there is at least one set of estimates of $\{p_i\}$ containing at least one estimate which is not a maximum likelihood value, such that the reciprocal of the determinant of its variances and covariances is greater than or equal to H. Let this set be $\{u_i\}$. Then, by hypothesis, $K \geq H$.

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[4] R. A. Fisher, The theory of statistical estimation, Proceedings of the Cambridge Philosophical Society, Vol. 22 (1925) pp. 700-725.

Let T be any linear transformation $u_i - p_i = \sum_{\alpha=1}^{n} a_{i\alpha} x_\alpha$ of pure rotation of the axes representing $u_i - p_i \; (i = 1, 2, \cdots n)$ about the point $p_1, p_2, \cdots p_n$ as origin, which will reduce $\sum_{i,j=1}^{n} k_{ij}(u_i - p_i)(u_j - p_j)$ to a sum of squares, $\sum_{\alpha=1}^{n} \bar{k}_\alpha x_\alpha^2$. Here we have $\bar{k}_\alpha = \sum_{i,j=1}^{n} k_{ij} a_{i\alpha} a_{j\alpha}$ and $x_\alpha = \sum_{i=1}^{n} b_{i\alpha}(u_i - p_j)$, where $b_{i\alpha}$ is the cofactor of $a_{i\alpha}$ in $|a_{ij}|$. Then $\bar{k}_\alpha$ is the reciprocal of the variance of the variable $\bar{u}_\alpha = \sum_{i=1}^{n} b_{i\alpha} u_i$ about its mean value $\bar{p}_\alpha = \sum_{i=1}^{n} b_{i\alpha} p_i$, and $\prod_{\alpha=1}^{n} \bar{k}_\alpha = K$, since the determinant $|a_{ij}|$ of T is unity. But $\bar{u}_\alpha$ is not the maximum likelihood estimate of $\bar{p}_\alpha$, since at least one of the u's is not a maximum likelihood value. As a matter of fact $\sum_{i=1}^{n} b_{i\alpha} t_i = \bar{t}_\alpha$, say, is the maximum likelihood value of $\bar{p}_\alpha$, for

$$\frac{\partial L}{\partial \bar{p}_\alpha} = \sum_{i=1}^{n} \frac{\partial L}{\partial p_i} \frac{\partial p_i}{\partial \bar{p}_\alpha} = \sum_{i=1}^{n} a_{i\alpha} \frac{\partial L}{\partial p_i}$$

vanishes only for $p_i = t_i$, that is, for $\bar{p}_\alpha = \sum_{i=1}^{n} b_{i\alpha} t_i$; (provided we assume that $\frac{\partial L}{\delta p_i} = 0 \; (i = 1, 2, \cdots n)$ has the unique solution $p_i = t_i$).

It follows from Fisher's[5] proof for the case of one variable, that the reciprocal q_α of the variance of $\bar{t}_\alpha$ is greater than $\bar{k}_\alpha$. Hence, $\prod_{\alpha=1}^{n} q_\alpha > \prod_{\alpha=1}^{n} \bar{k}_\alpha$. We note however, that the maximum likelihood estimates $\{\bar{t}_\alpha\}$ are not independent, for their distribution is

$$\frac{|a_{ij}| \sqrt{H}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2} \sum_{\alpha,\beta=1}^{n} \bar{h}_{\alpha\beta} (\bar{t}_\alpha - \bar{p}_\alpha)(\bar{t}_\beta - \bar{p}_\beta)}$$

where $\bar{h}_{\alpha\beta} = \sum_{i,j=1}^{n} h_{ij} a_{i\alpha} a_{j\beta}$, which is not necessarily zero for $\alpha \neq \beta$. The effect of this non-independence is to introduce a term $\frac{1}{R}$ as a multiplier of $\prod_{\alpha=1}^{n} q_\alpha$, where R is the determinant of correlations among the $\bar{t}$'s, and is less than unity. Hence, $\prod_{\alpha=1}^{n} \frac{q_\alpha}{R} > \prod_{\alpha=1}^{n} q_\alpha >$

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[5] R. A. Fisher, loc. cit.

$\prod_{\alpha=1}^{n}\bar{k}_\alpha = K$. It is well known from the theory of quadratic forms that the matrix $\|\bar{h}_{\alpha\beta}\|$ is found as the product $\|\bar{a}_{ij}\|\cdot\|h_{ij}\|\cdot\|a_{ij}\|$, where $\|\bar{a}_{ij}\|$ is $\|a_{ij}\|$ with its rows and columns interchanged. Since the determinant $|a_{ij}|$ is unity, it is clear that $|\bar{h}_{\alpha\beta}|$, which is equal to $\prod_{\alpha=1}^{n}\frac{q_\alpha}{k}$, has the value $|h_{ij}|$ which is H by definition. Therefore we have $H>K$, which contradicts the hypothesis that $K \geq H$. Hence, we must have $K<H$.

Thus, the proposition is proved that the reciprocal of the determinant of variances and covariances of the maximum likelihood estimates $\{t_i\}$ is smaller than that of any other set of estimates, all of which are not likelihood values.

We are now provided with a means of measuring the joint efficiency of a set of estimates in utilizing information in the sample relevant to the population parameters estimated by the set.. We shall take as a measure of this efficiency the ratio of the reciprocal of the determinant of its variances and covariances to that of the set of maximum likelihood estimates of the same parameters. This quantity is less than unity, as we have just proved. The efficiency of $\{u_i\}$ is therefore

(4)
$$Eff = \frac{K}{H}.$$

2. Simultaneous estimation of a and b.

We shall suppose that satisfactory estimates have been obtained for σ_x, σ_y and r. If they are to be taken from the sample ω, we can take σ_x^2 as the variance of the x's in ω_{xy} and ω_x, σ_y^2 as that of the y's in ω_{xy} and ω_y and r from ω_{xy}. In any case our problem is to find the optimum values of a and b for given values of σ_x, σ_y and r. These values of a and b are found as the solution of the equations obtained by setting (e_1) and (e_2) in (2) equal to zero. Accordingly, we find,

$$\hat{a} = \frac{1}{\Delta\sigma_y}\left[\frac{(1+\beta)\bar{x}}{\sigma_x(1-r^2)} + \frac{\alpha\bar{x}_1}{\sigma_x}\left(\frac{1}{1-r^2}+\beta\right) + \frac{\beta r}{\sigma_x(1-r^2)}(\bar{y}_1-\bar{y})\right]$$

(5)

$$\hat{b} = \frac{1}{\Delta\sigma_x}\left[\frac{(1+\alpha)\bar{y}}{\sigma_y(1-r^2)} + \frac{\beta\bar{y}_1}{\sigma_y}\left(\frac{1}{1-r^2}+\alpha\right) + \frac{\alpha r}{\sigma_y(1-r^2)}(\bar{x}_1-\bar{x})\right]$$

where $\Delta = \frac{1}{\sigma_x\sigma_y(1-r^2)}\left[1+\alpha+\beta+\alpha\beta(1-r^2)\right]$.

The matrix of the variances and covariances of $\hat{a}$ and $\hat{b}$ in samples is obtained by taking the reciprocal form of the two way principal minor in the upper left corner of the matrix (2)[6] Thus we find,

$$(6) \qquad \left\|\begin{array}{cc} \frac{\sigma_x^2[1+\beta(1-r^2)]}{sD} & \frac{r\sigma_x\sigma_y}{sD} \\ \frac{r\sigma_x\sigma_y}{sD} & \frac{\sigma_y^2[1+\alpha(1-r^2)]}{sD} \end{array}\right\|$$

where $D = 1+\alpha+\beta+\alpha\beta(1-r^2)$

We note from (6) that the variance of $\hat{a}$ is

$$\sigma_{\hat{a}}^2 = \frac{\sigma_x^2[1+\beta(1-r^2)]}{s[1+\alpha+\beta+\alpha\beta(1-r^2)]},$$

and a similar expression holds for $\sigma_{\hat{b}}^2$. The correlation coefficient of $\hat{a}$ and $\hat{b}$ is

$$r_{\hat{a}\hat{b}} = \frac{r}{\{[1+\beta(1-r^2)][1+\alpha(1-r^2)]\}^{\frac{1}{2}}}.$$

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[6] See Karl Pearson, On the influence of natural selection on the variability and correlation of organs, Philosophical Transactions of the Royal Society of London, series A, vol. 200 (1900), pp. 3-10. Here Pearson gives a method of obtaining the variances and co-variances of the variates in a normal multivariate probability function.

From the definitions in section 1, we find that the amount of information in ω relative to a and b is the reciprocal of the determinant of (5). That is,

$$A(m,n,s)=\frac{s^2+s(m+n)+mn(1-r^2)}{\sigma_x^2\sigma_y^2(1-r^2)} \quad (7)$$

From (7) we can find the relative amounts of information contributed by members of ω_{xy}, ω_x and ω_y by means of differences. For given values of n and s, we have as the information contributed to ω by an $m+1st$ individual of ω_x,

$$A_{\omega_x}(m+1)=A(m+1,n,s)-A(m,n,s)=\frac{s+n(1-r^2)}{\sigma_x^2\sigma_y^2(1-r^2)}, \quad (7a)$$

which is independent of m. A similar expression holds for the $n+1st$ member of ω_y.

For the $s+1st$ member of ω_{xy}, for given values of m and n, we get

$$A_{\omega_{xy}}(s+1)=\frac{m+n+2s+1}{\sigma_x^2\sigma_y^2(1-r^2)}. \quad (7b)$$

It is clear that an additional member to ω_{xy} is more informative than one to each of ω_x and ω_y by an amount $\frac{(m+n+1)r^2}{\sigma_x^2\sigma_y^2(1-r^2)}$, or, considering the ratio rather than the difference, we have

$$\frac{A(m+1,n+1,s)-A(m,n,s)}{A_{\omega_{xy}}(s+1)}=1-\frac{r^2(m+n+1)}{2s+m+n+1)}. \quad (7c)$$

We find that the amount of information introduced by ω_x and ω_y is $A(m,n,s)-A(0,0,s)$, which is $\frac{s(m+n)+mn(1-r^2)}{\sigma_x^2\sigma_y^2(1-r^2)}$ and its ratio to the total information (7) is

$$1-\frac{s^2}{s^2+s(m+n)+mn(1-r^2)}.$$

3. Simultaneous estimation of σ_x and σ_y.

If we suppose that r is given as well as a and b, we can find the optimum value of σ_x^2 and σ_y^2 by solving the equations obtained by setting (e_3) and (e_4) in (2) equal to zero. Accordingly, we get,

$$(8)\quad \hat{\sigma}_x^2 = \frac{2E(EF-G^2)}{2EF(1+\alpha)-G^2(\alpha-\beta)+\sqrt{4EFG^2(1+\alpha)(1+\beta)+G^4(\alpha-\beta)^2}}$$

$$\hat{\sigma}_y^2 = \frac{}{2EF(1+\beta)-G^2(\beta-\alpha)+\sqrt{4EFG^2(1+\alpha)(1+\beta)+G^4(\alpha-\beta)^2}}$$

where $E = \alpha\bar{u} + \frac{\bar{\xi}}{1-r^2}$, $F = \beta\bar{v} + \frac{\bar{\eta}}{1-r^2}$ and $G = \frac{r\bar{\zeta}}{1-r^2}$.

The variances and covariances of $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are given by the reciprocal form of the matrix obtained by striking out the last row and column from the third order principal minor in the lower right corner of (3). For the variance of $\hat{\sigma}_x$, we find,

$$\sigma^2_{\hat{\sigma}_x} = \frac{\sigma_x^2\,[2(1+\beta)-r^2(2\beta+1)]}{2s\,[2(1+\alpha)(1+\beta)-r^2(\alpha+\beta+\alpha\beta)]} .$$

A similar expression exists for $\sigma^2_{\sigma_y}$. The amount of information yielded by ω relative to σ_x and σ_y under these conditions is

$$(9)\quad A'(m,n,s) = \frac{4(m+s)(n+s)-2r^2[sm+sn+2mn]}{\sigma_x^2\sigma_y^2(1-r^2)} .$$

From (9), we find the differences corresponding to (7,a,b,c) to be

$$(9a)\quad A'_{\omega_x}(m+1) = \frac{4(s+n)-2r^2(s+2n)}{\sigma_x^2\sigma_y^2(1-r^2)} ,$$

$$(9b)\quad A'_{\omega_{xy}}(s+1) = \frac{8s+4(m+n+1)-2r^2(m+n)}{\sigma_x^2\sigma_y^2(1-r^2)} ,$$

$$(9c)\quad \frac{A'(m+1,n+1,s)-A(m,n,s)}{A'_{\omega_{xy}}(s+1)} = 1 - \frac{r^2(m+n+2s+2)}{4s+2(m+n+1)-r^2(m+n)} .$$

4. Simultaneous estimation of σ_x, σ_y and r.

Let us suppose a and b to be satisfactorily estimated. For large samples, a and b can be estimated from the sets of x's and y's obtained by pooling ω_{xy}, ω_x and ω_y. Whatever estimates we may choose for a and b, our problem is to solve the equations obtained by setting (e_3), (e_4) and (e_5) in (2) equal to zero, for σ_x, σ_y and r.

If we denote the quantities in the brackets of (e_3), (e_4) and (e_5) by f, g and h respectively, then we are to solve the equations $f=g=h=0$ for σ_x, σ_y and r. The method of elimination seems to be of little value in solving these equations. Then we shall use the extended form of Newton's approximation method and find an approximate solution. Considering nothing higher than the first order terms of Taylor's expansion of f, g and h we have (letting $\sigma_x=x$, $\sigma_y=y$, $r=z$),

$$(10)\quad \left\{\begin{array}{l} f_1+(x-x_1)f_{x_1}+(y-y_1)f_{y_1}+(z-z_1)f_{z_1}=0 \\ g_1+(x-x_1)g_{x_1}+(y-y_1)g_{y_1}+(z-z_1)g_{z_1}=0 \\ h_1+(x-x_1)h_{x_1}+(y-y_1)h_{y_1}+(z-z_1)h_{z_1}=0 \end{array}\right]$$

where $f_1=f(x_1,y_1,z_1)$, $f_{x_1}=\dfrac{\partial f(x_1,y_1,z_1)}{\partial x}$ and so on.

We shall take for the initial point, $x_1=\sqrt{\bar{\xi}}$, $y_1=\sqrt{\bar{\eta}}$ and $z_1=\dfrac{\bar{\zeta}}{\sqrt{\bar{\xi}\bar{\eta}}}$, which are, for $m=n=0$ and $a=\bar{x}$, $b=\bar{y}$, the maximum likelihood estimates of σ_x, σ_y and r from ω.

Solving equations (10) for x, y and z by Cramer's rule we find for the first approximation beyond the initial values,

$$(11)\quad \begin{aligned} \tilde{\sigma}_x &= \sqrt{\bar{\xi}}\left[1+\frac{\alpha\left(\frac{\bar{u}-\bar{\xi}}{\bar{\xi}}\right)\left(1+\frac{\beta\bar{v}}{\bar{\eta}}(1-\rho^4)\right)+\rho^2\beta\left(\frac{\bar{v}-\bar{\eta}}{\bar{\eta}}\right)}{2D}\right] \\ \tilde{\sigma}_y &= \sqrt{\bar{\eta}}\left[1+\frac{\beta\left(\frac{\bar{v}-\bar{\eta}}{\bar{\eta}}\right)\left(1+\frac{\alpha\bar{u}}{\bar{\eta}}(1-\rho^4)\right)+\rho^2\alpha\left(\frac{\bar{u}-\bar{\xi}}{\bar{\xi}}\right)}{2D}\right] \\ \tilde{r} &= \rho\left[1+(1-\rho^2)^2\,\frac{\alpha\left(\frac{\bar{u}-\bar{\xi}}{\bar{\xi}}\right)\left(\frac{1}{1-\rho^2}+\frac{\beta\bar{v}}{\bar{\eta}}\right)+\beta\left(\frac{\bar{v}-\bar{\eta}}{\bar{\eta}}\right)\left(\frac{1}{1-\rho^2}+\frac{\alpha\bar{u}}{\bar{\xi}}\right)}{2D}\right] \end{aligned}$$

where $D = 1 + \left(\frac{\beta\bar{v}}{\bar{\eta}} + \frac{\alpha\bar{u}}{\bar{\xi}}\right) + \frac{\alpha\beta\bar{u}\bar{v}}{\bar{\xi}\bar{\eta}}(1-\rho^4)$ and $\rho = \frac{\bar{\zeta}}{\sqrt{\bar{\xi}\bar{\eta}}}$. By using the point whose coordinates are given by (11) in place of the initial point in (10), we find a second approximation point, and continuing the process we get a sequence of points. Such a sequence would raise questions of convergence which will not be considered in this paper. However, it can be shown without much difficulty that the likelihood of the point whose coordinates are given by (11) is greater than that of the initial point for variations of $\bar{u}$ and $\bar{v}$ about $\bar{\xi}$ and $\bar{\eta}$ respectively, and for $\bar{u} = \bar{\xi}$ and $\bar{v} = \bar{\eta}$ the likelihoods are equal. Indeed the problem is equivalent to showing that the ratio of the likelihood (1) with the values $\sqrt{\bar{\xi}}$, $\sqrt{\bar{\eta}}$ and $\frac{\bar{\zeta}}{\sqrt{\bar{\xi}\bar{\eta}}}$ for σ_x, σ_y and r to the likelihood with the values given by (11) for σ_x, σ_y and r has a maximum of unity for variations of $\bar{u}$ and $\bar{v}$ about $\bar{\xi}$ and $\bar{\eta}$. This can be readily done by examining, in the ordinary manner for maxima and minima, the first and second derivatives with respect to $\bar{u}$ and $\bar{v}$ of the ratio of these likelihoods.

The matrix of limiting values of the sampling variances and covariances of the maximum likelihood estimates of σ_x, σ_y and r can be obtained by taking the reciprocal form of the third order principal minor in the lower right hand corner of (3). This reciprocal matrix is,

(12)

$$\left\| \begin{matrix} \frac{\sigma_x^2(1+\beta(1-r^4))}{2sE} & \frac{r^2\sigma_x\sigma_y(1-r^2)}{2sE} & \frac{r\sigma_x(1-r^2)(1+\beta(1-r^2))}{2sE} \\ \frac{r^2\sigma_x\sigma_y(1-r^2)}{2sE} & \frac{\sigma_y^2(1+\alpha(1-r^4))}{2sE} & \frac{r\sigma_y(1-r^2)(1+\alpha(1-r^2))}{2sE} \\ \frac{r\sigma_x(1-r^2)(1+\beta(1-r^2))}{2sE} & \frac{r\sigma_y(1-r^2)(1+\alpha(1-r^2))}{2sE} & \frac{(1-r^2)[1+(\alpha+\beta)(1-\frac{r^2}{2})+\alpha\beta(1-r^4)]}{2sE} \end{matrix} \right\|$$

where $E = 1+\alpha+\beta+\alpha\beta(1-r^4)$.

The amount of information in ω relative to σ_x, σ_y and r is the reciprocal of the determinant of (12). Denoting this quantity by $B(m, n, s)$, we have.

$$(13)\quad B(m,n,s)=\frac{4}{\sigma_x^2\sigma_y^2(1-r^2)^3}\left[s^3+s^2(m+n)+mns(1-r^4)\right].$$

Proceeding as we did with (7a), (7b) and (7c), we find the following incremental contributions:

$$(13a)\quad B_{\omega_x}(m+1)=\frac{4}{\sigma_x^2\sigma_y^2(1-r^2)^3}\left[s^2+sn(1-r^4)\right]$$

$$(13b)\quad B_{\omega_{xy}}(s+1)=\frac{4}{\sigma_x^2\sigma_y^2(1-r^2)^3}\left[3s^2+3s+1+(2s+1)(m+n)+mn(1-r^4)\right].$$

$$(13c)\quad B(m+1,n+1,s)-B(m,n,s)=\frac{4}{\sigma_x^2\sigma_y^2(1-r^2)^3}\left[2s^2+s(m+n+1)(1-r^4)\right]$$

$$(13d)\quad B(m,n,s)-B(0,0,s)=\frac{4}{\sigma_x^2\sigma_y^2(1-r^2)^3}\left[s^2(m+n)+mns(1-r^4)\right]$$

We note that the $s+1st$ member of ω_{xy} is much more important than an additional item to each of ω_x and ω_y when σ_x, σ_y and r are considered, than when a and b are considered. The amount of information contributed relative to r by ω_x and ω_y can be found by differencing the reciprocal of the element in the lower right corner of (12), with respect to m and n. If we call this reciprocal $K_{re}(m,n,s)$, we have as the ratio of the contribution of information by ω_x and ω_y, to the total information in ω regarding r,

$$\frac{K_{re}(m,n,s)-K_{re}(0,0,s)}{K_{re}(m,n,s)}=\frac{\frac{r^2}{2}s(m+n)+r\,mn(1-r^2)}{s^2+s(m+n)+mn(1-r^4)}.$$

In a similar manner, we can find the contributions of the various parts ω_{xy}, ω_x and ω_y to the information relative to any one of the parameters σ_x, σ_y and r and we can find their effects upon the covariances of the maximum likelihood estimates by considering the non-diagonal elements of (12). We find that the information afforded by ω_y relative to σ_x expressed in terms of the total amount of information in ω_{xy}, ω_x and ω_y regarding σ_x is,

$$\frac{r^4 sn}{s^2+s(m+n)+mn(1-r^4)}.$$

We remark without going further, that, by considering the five equations obtained by equating each of the expressions in (2) to zero, we can find approximations for the maximum likelihood estimates of a, b, σ_x, σ_y and r by the foregoing method Since the process is straightforward, though somewhat cumbersome in that it involves fifth order determinants, we shall not consider it here.

III. Systems of independent estimates.

We have seen that the problem of finding the maximum likelihood estimates of a, b, σ_x, σ_y and r from the sample leads to expressions which are not very simple, especially from the point of view of practical application. However, the variances and covariances of these estimates were found to be relatively simple. In view of the difficulties connected with the foregoing maximum likelihood estimates, we shall devote the remainder of this paper to a consideration of the moments, distributions and efficiencies of simpler systems of estimates.

If we are interested in the means of the x's in ω apart from any contribution of the y's, the optimum value of a is $\bar{x}_0 = \frac{\bar{x}+\alpha\bar{x}_1}{1+\alpha}$. Similarly, for the means of the y's, we have $\bar{y}_0 = \frac{\bar{y}+\beta\bar{y}_1}{1+\beta}$. The best estimates of the variances σ_x^2, and σ_y^2 under these conditions are,

$$\xi_0 = \frac{1}{N_1}\left[s\xi + mu + s(\bar{x}-\bar{x}_0)^2 + m(\bar{x}_1-\bar{x}_0)^2\right]$$

$$\eta_0 = \frac{1}{N_2}\left[s\eta + nv + s(\bar{y}-\bar{y}_0)^2 + n(\bar{y}_1-\bar{y}_0)^2\right],$$

where $N_1 = s+m$, and $N_2 = s+n$.

For the covariance we shall take the product moment of the deviations of the x's and y's in ω_{xy} from $\bar{x}_0$ and $\bar{y}_0$ respectively. That is,

$$\zeta_0 = \frac{1}{s}\sum_{i=1}^{s}(x_i-\bar{x}_0)(y_i-\bar{y}_0) = \zeta + (\bar{x}-\bar{x}_0)(\bar{y}-\bar{y}_0).$$

From these values, we can take as the estimate of the correlation coefficient,

$$r_0 = \frac{\zeta_0}{\sqrt{\xi_0\,\eta_0}}.$$

1. Distribution of $\bar{x}_0$ and $\bar{y}_0$

The variances and correlation of $\bar{x}_0$ and $\bar{y}_0$ can be found from

$$\frac{\sqrt{mns}}{(2\pi)^2\sigma_x^2\sigma_y^2\sqrt{1-r^2}}\, e^{-\frac{s}{2(1-r^2)}\left[\frac{(\bar{x}-a)^2}{\sigma_x^2}+\frac{(\bar{y}-b)^2}{\sigma_y^2}-\frac{2r(\bar{x}-a)(\bar{y}-b)}{\sigma_x\sigma_y}\right]-\frac{m}{2\sigma_x^2}(\bar{x}_1-a)^2-\frac{n}{2\sigma_y^2}(\bar{y}_1-b)^2}$$

by making the substitution $\bar{x}_0 = \frac{\bar{x}+\alpha\bar{x}_1}{1+\alpha}$, $\bar{y}_0 = \frac{\bar{y}+\beta\bar{y}_1}{1+\beta}$, and using determinant analysis[7] on the symmetric matrix of the resulting quadratic exponential.

The variances of $\bar{x}_0$ and $\bar{y}_0$ are found to be $\frac{\sigma_x^2}{N_1}$ and $\frac{\sigma_y^2}{N_2}$ respectively, and the correlation between $\bar{x}_0$ and $\bar{y}_0$ is $\frac{rs}{\sqrt{N_1N_2}}$. The exact distribution of $\bar{x}_0$ and $\bar{y}_0$ is normal.

The amount of information relative to a and b furnished by $\bar{x}_0$ and $\bar{y}_0$ is, according to our definition,

$$\frac{(s+m)(s+n)}{\sigma_x^2\sigma_y^2\left[1-\frac{r^2s^2}{(m+s)(n+s)}\right]}. \tag{14}$$

The efficiency of x_0 and y_0 is, therefore, the ratio of (14) to (7), which is

$$\frac{(m+s)^2(n+s)^2(1-r^2)}{[(m+s)(n+s)-mnr^2][(m+s)(n+s)-s^2r^2]}$$

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[7] Karl Pearson, loc cit.

2. Characteristic function of ξ_0, η_0 and ζ_0.

The characteristic function or generating function of the moments of ξ_0, η_0 and ζ_0, which we shall denote by $\phi(\gamma,\delta,\varepsilon)$, is defined as the mathematical expectation of $e^{\gamma\xi_0+\delta\eta_0+\varepsilon\zeta_0}$. Since ξ_0, η_0 and ζ_0 are expressible in terms of $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$, u, v, ξ, η and ζ, whose distribution is given by (1), then clearly, we can write,

$$(15) \qquad \phi(\gamma,\delta,\varepsilon) = \int e^{\gamma\xi_0+\delta\eta_0+\varepsilon\zeta_0} F\, dV,$$

Where F is given by (1) and dV is the product of the differentials of the variables in F, and the integration in taken over all possible values of the variables.

The integral (15) can be broken into the product of a constant by a quadruple integral, a triple integral and two single integrals. The quadruple integral is of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sum_{i,j=1}^{4} b_{ij}(t_i-c_i)(t_j-c_j)}\, dt_1\, dt_2\, dt_3\, dt_4$$

which has the value[8] $\frac{\pi^2}{\sqrt{\Delta}}$, where Δ is the determinant $|b_{ij}|$ ($i,j=1,2,3,4$) and $b_{ij}=b_{ji}$. The triple integral is of the form

$$\int_0^{\infty}\int_0^{\infty}\int_{-\sqrt{xy}}^{\sqrt{xy}} e^{-(c_{11}x+c_{22}y+c_{12}z)} (xy-z^2)^{\frac{s-4}{2}}\, dz\, dx\, dy$$

which has the value[9]

$$\frac{\sqrt{\pi}\,\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s-2}{2}\right)}{\begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix}^{\frac{s-1}{2}}}$$

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[8] Karl Pearson, loc. cit.

[9] See V. Romanovsky, On the moments of the standard deviations and of the correlation coefficient in samples from a normal population, Metron, vol. 5, no. 4 (1925) pp. 3-46.

Each of the single integrals is of the well known form

$$\int_0^\infty t^{k-1} c^{-c^2 t}\, dt$$

which has the value $\frac{\Gamma(K)}{C^{2K}}$

Using the above results for the integrals into which (15) resolves itself, we get.

(16)
$$\phi(\gamma,\delta,\varepsilon) = A^{\frac{m}{2}} B^{\frac{n}{2}} (\bar{A}\bar{B}-C^2)^{\frac{s-1}{2}} \left(A-\frac{\gamma}{N_1}\right)^{-\frac{m-1}{2}} \left(b\ \frac{\delta}{N_2}\right)^{-\frac{n-1}{2}}$$

$$\times\left[\left(\bar{A}-\frac{\gamma}{N_1}\right)\left(\bar{B}-\frac{\delta}{N_2}\right)-\left(C+\frac{\varepsilon}{2s}\right)^2\right]^{-\frac{s-1}{2}} \left[\left(A-\frac{\gamma}{N_1}\right)\left(B-\frac{\delta}{N_2}\right)\right.$$

$$\left.+\frac{\varepsilon^2}{4s^2}\left(\frac{r^2m^2n^2}{N_1^2N_2^2}-\frac{mn}{N_1N_2}\right)-\frac{\varepsilon r mn}{2s\sigma_x\sigma_y N_1 N_2}-\frac{\gamma\delta\, mnr^2}{N_1^2N_2^2}\right]^{-\frac{1}{2}},$$

where $A=\frac{1}{2\sigma_x^2}$, $B=\frac{1}{2\sigma_y^2}$, $\bar{A}=\frac{1}{2\sigma_x^2(1-r^2)}$, $\bar{B}=\frac{1}{2\sigma_y^2(1-r^2)}$,

$$C=\frac{r}{2\sigma_x\sigma_y(1-r^2)}.$$

If we write $M(h,k,l)=\frac{\partial^h}{\partial\gamma^h}\frac{\partial^k}{\partial\delta^k}\frac{\partial^l}{\partial\varepsilon^l}\ \phi(\gamma,\delta,\varepsilon)\Big|_{\gamma=\delta=\varepsilon=0}$

we find the following expressions for the first few moments of ξ_0, η_0 and ζ_0,

$$M(1,0,0)=\frac{N_1-1}{N_1}\sigma_x^2, \quad M(0,1,0)=\frac{N_2-1}{N_2}\sigma_y^2$$

$$M(0,0,1)=r\sigma_x\sigma_y\left(\frac{s-1}{s}+\frac{mn}{sN_1N_2}\right)$$

$$M(1,1,0)=\frac{\sigma_x^2\sigma_y^2}{N_1N_2}\left[(N_1-1)(N_2-1)+2r^2\left(\frac{mn}{N_1N_2}+s-1\right)\right]$$

(17) $$M(1,0,1)=\frac{r\sigma_x^3\sigma_y}{N_1 s}(N_1+1)\left(\frac{mn}{N_1N_2}+s-1\right)$$

$$M(0,1,1)=\frac{r\sigma_x\sigma_y^{3}}{N_2\, s}(N_2+1)\left(\frac{mn}{N_1N_2}+s-1\right)$$

$$M(2,0,0)=\frac{\sigma_x^{4}(N_1^{2}-1)}{N_1^{2}},\quad M(0,2,0)=\frac{\sigma_y^{4}(N_2^{2}-1)}{N_2^{2}},$$

$$M(0,0,2)=\frac{\sigma_x^{2}\sigma_y^{2}}{s^{2}}\left[(1+r^{2})(s-1)+r^{2}\frac{m^{2}n^{2}}{N_1^{2}N_2^{2}}+\frac{mn}{N_1N_2}+r^{2}\left(\frac{mn}{N_1N_2}+s-1\right)^{2}\right]$$

$$M(1,1,1)=\frac{r\sigma_x^{3}\sigma_y^{3}}{N_1N_2}\left[\frac{4(s-1)(1+r^{2})}{s}+\frac{4mn}{N_1N_2}\left(r^{2}\frac{mn}{N_1N_2}+1\right)\right.$$

$$\left.+\frac{2r^{2}}{s}\left(\frac{mn}{N_1N_2}+s-1\right)^{2}+\left(\frac{mn}{sN_1N_2}+\frac{s-1}{s}\right)\left(N_1N_2+N_1+N_2-3\right)\right].$$

If the sample ω is fairly large, we can neglect the contributions of the means $\bar{x}$, $\bar{y}$, $\bar{x}_1$, and $\bar{y}_1$ to ξ_0, η_0, $\mathfrak{z}_0$ and consider as satisfactory estimates of the variances and covariance,

$$\bar{\xi}_0=\frac{\xi+\alpha u}{1+\alpha},\qquad \bar{\eta}_0=\frac{\eta+\beta v}{1+\beta},\qquad \bar{\mathfrak{z}}_0=\mathfrak{z}\,.$$

3. Characteristic function and sampling distribution of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\mathfrak{z}}_0$.

It is clear that $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\mathfrak{z}}_0$ are obtained from ξ_0, η_0 and $\mathfrak{z}_0$ by dropping the terms involving the means $\bar{x}$, $\bar{y}$, $\bar{x}_1$ and $\bar{y}_1$. The characteristic function $\bar{\varphi}(\gamma,\delta,\varepsilon)$ of these statistics can be obtained from (15) by replacing ξ_0, η_0 and $\mathfrak{z}_0$ by $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\mathfrak{z}}_0$ and integrating. The integral in this case will not involve the quadruple integral, but only the triple integral and the two single integrals. Accordingly, we find

$$\bar{\varphi}(\gamma,\delta,\varepsilon)=A^{\frac{m-1}{2}}B^{\frac{n-1}{2}}(\bar{A}\bar{B}-C^{2})^{\frac{s-1}{2}}\left(A-\frac{\gamma}{N_1}\right)^{-\frac{m-1}{2}}$$

$$\times\left(B-\frac{\delta}{N_2}\right)^{-\frac{n-1}{2}}\left[\left(\bar{A}-\frac{\gamma}{N_1}\right)\left(\bar{B}-\frac{\delta}{N_2}\right)-\left(C+\frac{\varepsilon}{2s}\right)^{2}\right]^{-\frac{s-1}{2}}$$

which is somewhat simpler than (16).

The first few moments of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\mathfrak{z}}_0$ evaluated from $\bar{\varphi}(\gamma,\delta,\varepsilon)$ are (using the notation of (17),

$$M(1,0,0)=\frac{\sigma_x^2(N_1-2)}{N_1},\quad M(0,1,0)=\frac{\sigma_y^2(N_2-2)}{N_2},$$

$$M(0,0,1)=\frac{s-1}{s}r\sigma_x\sigma_y$$

$$M(1,1,0)=\frac{\sigma_x^2\sigma_y^2}{N_1N_2}\left[2r^2(s-1)+(N_1-2)(N_2-2)\right]$$

$$M(1,0,1)=\frac{r\sigma_x^3\sigma_y(s-1)}{s},\quad M(0,1,1)=\frac{r\sigma_x\sigma_y^3(s-1)}{s}$$

$$(18)\quad M(2,0,0)=\frac{\sigma_x^4(N_1-2)}{N_1},\qquad M(0,2,0)=\frac{\sigma_y^4(N_2-2)}{N_2}$$

$$M(0,0,2)=\frac{\sigma_x^2\sigma_y^2(s-1)(1+r^2s)}{s},$$

$$M(1,1,1)=\frac{(s-1)r\sigma_x^3\sigma_y^3}{N_1N_2s}\left[2(N_1+N_2-2)+2(s+1)r^2+(N_1-2)(N_2-2)\right].$$

In order to find the exact sampling distribution of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\zeta}_0$, it is more convenient to consider the statistics $\xi_1=\frac{N_1}{s}\bar{\xi}_0$, $\eta_0=\frac{N_2}{s}\bar{\eta}_0$ and $\zeta_1=\bar{\zeta}_0$. The characteristic function of these statistics is found from $\phi(\gamma,\delta,\varepsilon)$ by replacing $\frac{N_1}{s}\gamma$ by γ_1, $\frac{N_2}{s}\delta$ by δ_1, and ε by ε_1. Thus, we have

$$(19)\quad \phi_1(\gamma_1,\delta_1,\varepsilon_1)=A_1^aB_1^b(\bar{A}_1\bar{B}_1-C_1^2)^c(A_1-\gamma_1)^{-a}(B_1-\delta_1)^{-b}\left[(\bar{A}_1-\gamma_1)(\bar{B}_1-\delta_1)-\left(C_1+\frac{\varepsilon_1}{2}\right)^2\right]^{-c}$$

where A_1, B_1, $\bar{A}_1$, $\bar{B}_1$ and C_1 are the constants A, B, $\bar{A}$, $\bar{B}$ and C each multiplied by s, and $a=\frac{m-1}{2}$, $b=\frac{n-1}{2}$ and $c=\frac{s-1}{2}$. The distribution $f(\xi_1,\eta_1,\zeta_1)$ of ξ_1, η_1 and ζ_1 is then, the solution of the integral equation,

$$(20)\quad \int_0^\infty\int_0^\infty\int_{-\sqrt{\xi_1\eta_1}}^{\sqrt{\xi_1\eta_1}}e^{\gamma_1\xi_1+\delta_1\eta_1+\varepsilon_1\zeta_1}f(\xi_1,\eta_1,\zeta_1)\,d\xi_1\,d\eta_1\,d\zeta_1=\phi_1(\gamma_1,\delta_1,\varepsilon_1).$$

We note from (19) that the factor $(A_1-\gamma_1\ )^{-a}$ can be written as $(\bar{A}_1-\gamma_1)^{-a}\left(1-\frac{r^2\bar{A}_1}{\bar{A}_1-\gamma_1}\right)^{-a}$ and likewise with respect to $(B_1-\delta_1)^{-b}$. For sufficiently small values of γ_1 and δ_1 these terms can be represented by series expansions. It will be convenient to rearrange the product of these two series in a power series in r^2. Expanding and arranging in this manner we get

$$(21)\quad \varphi_1(\gamma_1,\delta_1,\varepsilon_1)=\frac{A_1^a B_1^b(\bar{A}_1\bar{B}_1-C_1^2)^c}{\Gamma(a)\Gamma(b)}(\bar{A}_1-\gamma_1)^{-a}(\bar{B}_1-\delta_1)^{-b}\left[(\bar{A}_1-\gamma_1)(\bar{B}_1-\delta_1)-(C_1+\tfrac{\varepsilon_1}{2})^2\right]^{-c}$$

$$\times\sum_{i=0}^{\infty}\frac{r^{2i}}{i!}\sum_{j=0}^{i}\binom{i}{j}\Gamma(a+i-j)\Gamma(b+j)\left(\frac{\bar{A}_1}{\bar{A}_1-\gamma_1}\right)^{i-j}\left(\frac{\bar{B}_1}{\bar{B}_1-\delta_1}\right)^{j}.$$

Each term of this expansion is of the form

$$(22)\quad \varphi_k(\gamma_1,\delta_1,\varepsilon_1)=G_k(\bar{A}_1-\gamma_1)^{-a_k}(\bar{B}_1-\delta_1)^{-b_k}\left[(\bar{A}_1-\gamma_1)(\bar{B}_1-\delta_1)-(C_1+\tfrac{\varepsilon_1}{2})^2\right]^{-c}$$

where G_k is a constant independent of γ_1, δ_1 and ε_1.

We are now in position to find $f(\xi_1,\eta_1,\zeta_1\ )$ as a series of terms $f_k(\xi_1,\eta_1,\zeta_1)$ whose form is given as the solution of

$$(23)\quad \int_0^{\infty}\int_0^{\infty}\int_{-\sqrt{\xi_1\eta_1}}^{\sqrt{\xi_1\eta_1}} e^{\gamma_1\xi_1+\delta_1\eta_1+\varepsilon_1\zeta_1} f_k(\xi_1,\eta_1,\zeta_1)\,d\xi_1\,d\eta_1\,d\zeta_1=\varphi_k(\gamma_1,\delta_1,\varepsilon_1)$$

The integral equation (23) can be solved by the methods used by Romanovsky[10] Following Romanovsky we find that

$$(24)\quad f_k(\xi_1,\eta_1,\zeta_1)=G_k e^{-\bar{A}_1\xi_1-\bar{B}_1\eta_1+2C_1\zeta_1}\,\xi_1^{a_k+c-\frac{3}{2}}\,\eta_1^{b_k+c-\frac{3}{2}}\,\omega\!\left(\frac{\zeta_1}{\sqrt{\xi_1\eta_1}}\right)$$

where $\omega\left(\frac{\zeta_1}{\sqrt{\xi_1\eta_1}}\right)$ is an even function of $\frac{\zeta_1}{\sqrt{\xi_1\eta_1}}=t$, say, which satisfies the condition

---

[10] V Romanovsky, loc. cit.

$$(25) \quad \int_{-1}^{+1} t^{2g}\omega(t)dt = \frac{\Gamma(g+\frac{1}{2})\Gamma(c+g)}{\Gamma(a_k+c+g)\Gamma(b_k+c+g)\Gamma(\frac{1}{2})\Gamma(c)} = M_g$$

for $g = 0, 1, 2, \ . \ . \ .$ and $\omega(t)$ is independent of g.

To solve (25) we observe that the right side can be written as

$$(26) \quad M_g = H\int_0^1\int_0^1 u^{g-\frac{1}{2}}(1-u)^{a_k+c-\frac{3}{2}} v^{c+g-1}(1-v)^{b_k-1} du\,dv,$$

where $H = \frac{1}{\Gamma(\frac{1}{2})\Gamma(c)\Gamma(b_k)\Gamma(a_k+c-\frac{1}{2})}$. The g-th moment of t^2 is now identical with the g-th moment of the product uv. Since ω(t) is even, we have,

$$(27) \qquad \int_0^1 t^{2g}\omega(t)dt = \frac{1}{2}M_g .$$

Setting $v = \frac{t^2}{u}$, $dv = \frac{2t}{u}dt$ in (26) we find

$$(28) \quad \omega(t) = H\int_{t^2}^1 (1-\frac{t^2}{u})^{a_k+c-\frac{3}{2}} u^{c-\frac{3}{2}}(1-u)^{b_k-1} du$$

Making the transformation $\frac{1-u}{1-t^2} = \theta$, we finally obtain,

$$(29) \quad \omega(t) = \frac{(1-t^2)^{a_k+b_k+c-\frac{3}{2}}}{\Gamma(\frac{1}{2})\Gamma(c)\Gamma(a_k+b_k+c-\frac{1}{2})} F\left[a_k, b_k, a_k+b_k+c-\frac{1}{2}, 1-t^2\right],$$

where the F function is the ordinary hypergeometric series. Using this form with t replaced by $\frac{\zeta_1}{\sqrt{\xi_1\eta_1}}$ in (24), we have $f_k(\xi_1, \eta_1, \zeta_1)$ fully determined.

The complete solution $f(\xi_1, \eta_1, \zeta_1)$ of (20) can be found by summing all of the expressions of the form (24) whose characteristic functions appear in the sum (21). Without much difficulty we can sum this series by expressing the coefficients as beta functions and interchanging the order of summation and integration. Accordingly, we can express $f(\xi_1, \eta_1, \zeta_1)$ in closed form as

$$(30)\quad f(\xi_1,\eta_1,\zeta_1)=\bar{K}e^{-\bar{A}_1\xi_1-\bar{B}_1\eta_1+2C_1\zeta_1}\xi_1^{a+c-\frac{3}{2}}\eta_1^{b+c-\frac{3}{2}}\left(1-\frac{\zeta_1^2}{\sqrt{\xi_1\eta_1}}\right)^{a+b+c-\frac{3}{2}}$$

$$\times\int_0^1\int_0^1(1-x)^{a+c-\frac{3}{2}}x^{b-1}\left[1-\left(1-\frac{\zeta_1^2}{\xi_1\eta_1}\right)x\right]^{-a}y^{a-1}(1-y)^{c-\frac{3}{2}}$$

$$\times e^{r^2\left(1-\frac{\zeta_1^2}{\xi_1\eta_1}\right)\left(\eta_1x_1\bar{B}_1+\frac{\xi_1(1-x)y\bar{A}_1}{1-\left(1-\frac{\zeta_1^2}{\xi_1\eta_1}\right)x}\right)}dx\,dy,$$

where $\bar{K}=\dfrac{A_1^aB_1^b(\bar{A}_1\bar{B}_1-C_1^2)^c}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c-\frac{1}{2})}$

The distribution of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\zeta}_0$ can be found by the change of variables $\xi_1=\frac{N_1}{s}\bar{\xi}_0$, $\eta_1=\frac{N_2}{s}\bar{\eta}_0$, $\zeta_1=\zeta_0$. It is clear that our estimate $t_0=\frac{\bar{\zeta}_0}{\sqrt{\bar{\xi}_0\bar{\eta}_0}}$ of the correlation coefficient can range in value from $-\sqrt{\frac{N_1N_2}{s^2}}$ to $\sqrt{\frac{N_1N_2}{s^2}}$.

4. Moments of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\zeta}_0$ when $r=0$.

The general product moment $M(h,k,t)=E(\bar{\xi}_0^h\bar{\eta}_0^k\bar{\zeta}_0^t)$ obtained from (30) for $r\neq 0$ is extremely unmanageable and impractical, since it is a generalized hypergeometric series expressed by a quadruple summation. However, for $r=0, M(h,k,t)$ is quite simple. Indeed, for this case, we have,

$$(31)\quad M(h,k,t)=\frac{\bar{K}\Gamma(a)\Gamma(c-\frac{1}{2})s^{h+k}}{\Gamma(a+c-\frac{1}{2})N_1^hN_2^k}\int_0^\infty\int_0^\infty\int_{-\sqrt{\xi_1\eta_1}}^{\sqrt{\xi_1\eta_1}}\int_0^1e^{-A_1\xi_1-B_1\eta_1}$$

$$\times\,\xi_1^{a+c+h-\frac{3}{2}}\eta_1^{b+c+k-\frac{3}{2}}\zeta_1^t\left(1-\frac{\zeta_1^2}{\xi_1\eta_1}\right)^{a+b+c-\frac{3}{2}}$$

$$\times(1-\theta)^{a+c-\frac{3}{2}}\theta^{b-1}\left[1-\left(1-\frac{\zeta_1^2}{\xi_1\eta_1}\right)\theta\right]^a d\theta\,d\zeta_1\,d\xi_1\,d\eta_1.$$

$M(h,k,t)$ exists for all positive values of h and k and for all positive-integral values of t. Since the integrand is an even function of ζ_1 it follows that $M(h, k, t) = 0$ for t an odd integer. If we let $t = 2\nu$, set $\zeta_1 = t\sqrt{\xi_1 \eta_1}$ in (31) and make use of (25), we find,

$$M(h,k,2\nu) = \frac{s^{h+k} A_1^{-h-\nu} B_1^{-k-\nu}}{N_1^h N_2^k} \tag{32}$$
$$\times \frac{\Gamma(b+c+k+\nu)\Gamma(a+c+h+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(c+\nu)}{\Gamma(b+c+\nu)\Gamma(a+c+\nu)\Gamma(c)\Gamma(\frac{1}{2})}.$$

The 2ν-th moment of the correlation coefficient can be found from (32) by letting $h = k = -\nu$. Thus,

$$M_{2\nu}(r_0) = \left(\frac{N_1 N_2}{s^2}\right)^{\nu} \frac{\Gamma(\nu+\frac{1}{2})\Gamma(c+\nu)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+c+\nu)\Gamma(b+c+\nu)\Gamma(c)\Gamma(\frac{1}{2})}.$$

The variance of r_0 is $\sigma_{r_0}^2 = \frac{(s-1)N_1 N_2}{s^2(N_1-2)(N_2-2)}$, which does not differ appreciably from $\frac{1}{s-1}$, which is the sampling variance of r when it is computed from ω_{xy}. The distribution of r_0 is found by setting $t = \frac{s}{\sqrt{N_1 N_2}} r_0$ in (29) and multiplying by $\Gamma(a+c)$ $\Gamma(b+c)$.

The 2ν-th moments of the regression coefficient of y on x, $\rho = \frac{\bar{\zeta}_0}{\bar{\xi}_0}$ say, is

$$M_{2\nu}(\rho) = M(-2\nu, 0, 2\nu) = \left(\frac{N_1}{s}\right)^{2\nu} \left(\frac{\sigma_y}{\sigma_x}\right)^{2\nu} \frac{\Gamma(a+c-\nu)\Gamma(\nu+\frac{1}{2})\Gamma(c+\nu)}{\Gamma(a+c+\nu)\Gamma(c)\Gamma(\frac{1}{2})},$$

and the variance is

$$\sigma_\rho^2 = \frac{N_1^2(s-1)}{(N_1-2)(N_1-4)s^2} \cdot \frac{\sigma_y^2}{\sigma_x^2}.$$

This differs very little from the variance $\frac{1}{s-3} \cdot \frac{\sigma_y^2}{\sigma_x^2}$ of the regression coefficient using only the data from ω_{xy}.

Slightly more accurate estimates can be obtained for σ_x^2, σ_y^2 and $r\sigma_x\sigma_y$ by multiplying $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\zeta}_0$ by $\frac{N_1}{N_1-2}$, $\frac{N_2}{N_2-2}$ and $\frac{s}{s-1}$ respectively. These corrected estimates will have their mathematical expectations identical with σ_x^2, σ_y^2 and $r\sigma_x\sigma_y$, as will be seen from $M(1,0,0)$, $M(0,1,0)$ and $M(0,0,1)$ in (18). In this case, the general moment $M(h,k,2\nu)$ will be identical with (32) multiplied by

$$\left(\frac{N_1}{N_1-2}\right)^h \left(\frac{N_2}{N_2-2}\right)^k \left(\frac{s}{s-1}\right)^{2\nu}$$

The variance for r_0 in this case is $\frac{1}{s-1}$, and that for the regression coefficient is

$$\frac{(N_1-2)\sigma_y^2}{(s-1)(N_1-4)\sigma_x^2}.$$

5. Variances and covariances of $\sqrt{\bar{\xi}_0}$, $\sqrt{\bar{\eta}_0}$ and r_0 in large samples.

As we have seen in the last section, the product moments of $\bar{\xi}_0$, $\bar{\eta}_0$ and $\bar{\zeta}_0$ evaluated from (30) are too complicated to be of much practical value, and there is not much hope from this source of finding the sampling variance of the estimate r_0 of the correlation coefficient The moments and variances of $\sqrt{\bar{\xi}_0}$ and $\sqrt{\bar{\eta}_0}$ taken separately are well known results. In fact, for large samples, the variances are $\frac{\sigma_x^2}{2N_1}$ and $\frac{\sigma_y^2}{2N_2}$ respectively. The variance of r_0 is not so immediately obtained. We shall find its limiting form for large samples from the normal form approached by the distribution of $\sqrt{\bar{\xi}_0}$, $\sqrt{\bar{\eta}_0}$ and r_0 as m, n and s approach ∞ in constant ratios $\frac{m}{\alpha}=\alpha$, $\frac{n}{\alpha}=\beta$.

For convenience let $\sqrt{\bar{\xi}_0}=\theta$, $\sqrt{\bar{\eta}_0}=\varphi$ and $r_0=t$. Then we have

$$(33)\quad \theta^2=\frac{\xi+\alpha u}{1+\alpha},\ \varphi^2=\frac{\eta+\beta v}{1+\beta},\ t=\frac{\zeta}{\theta\varphi}.$$

If we integrate (1) with respect to $\bar{x}$, $\bar{y}$, $\bar{x}_1$, and $\bar{y}_1$ and perform the following transformations on the remaining part of the distribution,

$$\xi=\theta^2(1+\alpha)-\alpha u \qquad d\xi=2\theta(1+\alpha)\,d\theta$$

$$\eta=\varphi^2(1+\beta)-\beta v \qquad d\eta=2\varphi(1+\beta)\,d\varphi$$

$$\zeta=t\theta\varphi \qquad d\zeta=\theta\varphi\,dt,$$

we can write it in the form,

$$(34)\qquad F(u,v,\theta,\phi,t)\left[f(u,v,\theta,\phi,t)\right]^{s},$$

where

$$F(u,v,\theta,\phi,t)=4\,C(1+\alpha)(1+\beta)(1-r^2)^{\frac{s}{2}}\sigma_x^{s+m}\sigma_y^{s+n}e^{-\frac{m+n+2s}{2}}$$
$$\times u^{-\frac{3}{2}}v^{-\frac{3}{2}}\theta^2\phi^2\left[(\overline{1+\alpha}\,\theta^2+\alpha u)(\overline{1+\beta}\,\phi^2-\beta v)-t^2\theta^2\phi^2\right]^{-2}$$

and

$$C=\frac{s^{s-1}m^{\frac{m-1}{2}}n^{\frac{n-1}{2}}\sigma_x^{-s-m+2}\sigma_y^{-s-n+2}(1-r^2)^{-\frac{s-1}{2}}}{2^{\frac{m+n+2s-4}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{s-1}{2})\Gamma(\frac{s-2}{2})\Gamma(\frac{m-1}{2})\Gamma(\frac{n-1}{2})}$$

and

$$f(u,v,\theta,\phi,t)=(1-r^2)^{-\frac{1}{2}}\sigma_x^{-1-\alpha}\sigma_y^{-1-\beta}e^{\frac{\alpha+\beta+2}{2}}u^{\frac{\alpha}{2}}v^{\frac{\beta}{2}}$$
$$\times\left[(\overline{1+\alpha}\,\theta^2-\alpha u)(\overline{1+\beta}\,\phi^2-\beta v)-t^2\theta^2\phi^2\right]^{\frac{1}{2}}$$
$$\times e^{-\frac{1}{2(1-r^2)}\left[\frac{\overline{1+\alpha}\,\theta^2-r^2\alpha u}{\sigma_x^2}+\frac{\overline{1+\beta}\,\phi^2-r^2\beta v}{\sigma_y^2}-\frac{2rt\theta\phi}{\sigma_x\sigma_y}\right]}.$$

If (34) were integrated with respect to u and v, we would get the distribution of $\theta=\sqrt{\frac{s}{N_1}}\xi_1$, $\phi=\sqrt{\frac{s}{N_2}}\eta_1$ and $t=\frac{\zeta_1}{\theta\phi}$ where the distribution of ξ_1, η_1 and ζ_1 is given by (30). The problem of finding the asymptotic normal form of the distribution of θ, ϕ and t from (30) seems extremely complicated. However, we can find this asymptotic form by first finding the limiting normal form of the distribution of u, v, θ, ϕ, and t from (34) and then integrating with respect to u and v.

The limiting normal form of (34) can be found by methods developed by von Mises[11] in a paper which appeared in 1919. In fact, $f(u,v,\theta,\phi,t)$ satisfies all of the conditions of the generalization of his first theorem to functions of more than one variable. In particular, the first order partial derivatives vanish and the determinant and all of its principal minors of the negative of the Hessian are positive at the point P whose coordinates are $u=\sigma_x^2$, $v=\sigma_y^2$, $\theta=\sigma_x$, $\phi=\sigma_y$ and $t=r$. Furthermore, f is identically zero outside the region of possible values of u, v, θ, ϕ, and t.

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[11] R. von Mises, Fundamentalsatze der Wahrscheinlichkeitsrechnung, Mathematische Zeitschrift, Bd 4 (1919) S. 14-18.

The matrix of the negative of the second derivatives at P is

$$
\begin{array}{c|c|c|c|c|c|}
 & \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial t} \\
\hline
\frac{\partial f}{\partial u} & \frac{\alpha}{2\sigma_x^4}\left(\frac{\alpha}{\rho^4}+1\right) & \frac{r^2\alpha\beta}{2\sigma_x^2\sigma_y^2\rho^4} & -\frac{\alpha}{\sigma_x^3\rho^2}\left(1+\frac{\alpha}{\rho^2}\right) & -\frac{\alpha\beta r^2}{\sigma_x^2\sigma_y\rho^4} & \frac{\alpha r}{\sigma_x^2\rho^4} \\
\hline
\frac{\partial f}{\partial v} & \frac{r^2\alpha\beta}{2\sigma_x^2\sigma_y^2\rho^4} & \frac{\beta}{2\sigma_y^4}\left(\frac{\beta}{\rho^4}+1\right) & -\frac{\alpha\beta r^2}{\sigma_x\sigma_y^2\rho^4} & -\frac{\beta}{\sigma_y^3\rho^2}\left(1+\frac{\beta}{\rho^2}\right) & \frac{\beta r}{\sigma_y^2\rho^4} \\
\hline
\frac{\partial f}{\partial \theta} & -\frac{\alpha}{\sigma_x^3\rho^2}\left(1+\frac{\alpha}{\rho^2}\right) & -\frac{\alpha\beta r^2}{\sigma_x\sigma_y^2\rho^4} & \frac{2}{\sigma_x^2\rho^2}\left(\frac{r^2}{2}+\frac{(\rho^2+\alpha)^2}{\rho^2}\right) & -\frac{2r^2}{\sigma_x\sigma_y\rho^2}\left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2}\right) & -\frac{2r}{\sigma_x\rho^2}\left(\frac{1}{2}+\frac{\alpha}{\rho^2}\right) \\
\hline
\frac{\partial f}{\partial \phi} & -\frac{\alpha\beta r^2}{\sigma_x^2\sigma_y\rho^4} & -\frac{\beta}{\sigma_y^3\rho^2}\left(1+\frac{\beta}{\rho^2}\right) & -\frac{2r^2}{\sigma_x\sigma_y\rho^2}\left(\frac{1}{2}-\frac{\alpha\beta}{\rho^2}\right) & \frac{2}{\sigma_y^2\rho^2}\left(\frac{r^2}{2}+\frac{(\rho^2+\beta)^2}{\rho^2}\right) & -\frac{2r}{\sigma_y\rho^2}\left(\frac{1}{2}+\frac{\beta}{\rho^2}\right) \\
\hline
\frac{\partial f}{\partial t} & \frac{\alpha r}{\sigma_x^2\rho^4} & \frac{\beta r}{\sigma_y^2\rho^4} & -\frac{2r}{\sigma_x\rho^2}\left(\frac{1}{2}+\frac{\alpha}{\rho^2}\right) & -\frac{2r}{\sigma_y\rho^2}\left(\frac{1}{2}+\frac{\beta}{\rho^2}\right) & \frac{1+r^2}{\rho^4} \\
\hline
\end{array}
$$

Now it follows at once from von Mises' theorem that

$$(36)\ F(u,v,\theta,\phi,t)\left[f(u,v,\theta,\phi,t)\right]^S \sim F(\sigma_x^2,\sigma_y^2,\sigma_x,\sigma_y,r)\,e^{-\frac{S}{2}\sum_{i,j=1}^{5} h_{ij}x_i x_j}$$

where $x_1=u-\sigma_x^2$; $x_2=v-\sigma_y^2$, $x_3=\theta-\sigma_x$, $x_4=\phi-\sigma_y$ and $x_5=t-r$, and h_{ij} is the element in the i-th row and j-th column of the matrix (35) Now,

$$F(\sigma_x^2,\sigma_y^2,\sigma_x,\sigma_y,r)=\frac{S^{\frac{5}{2}}\sqrt{\alpha\beta}\,(1+\alpha)(1+\beta)}{(2\pi)^{\frac{5}{2}}\sigma_x^3\sigma_y^3(1-r^2)^{\frac{3}{2}}}$$

which is equal to $\left(\frac{s}{2\pi}\right)^{\frac{5}{2}}\sqrt{h}$, where h is the determinant $|h_{ij}|$.

The variables in which we are primarily interested are θ, ϕ and t. The matrix of variances and covariances of θ, ϕ and t is formed by taking the third order matrix in the lower right corner of the reciprocal form of $\| h_{ij} \|$. This matrix turns out to be

$$(37)\qquad \begin{array}{c|c|c|c|}
 & \theta & \phi & t \\
\hline
\theta & \frac{\sigma_x^2}{2s(1+\alpha)} & \frac{r^2\sigma_x\sigma_y}{2s(1+\alpha)(1+\beta)} & \frac{r\sigma_x(1+\beta-r^2)}{2s(1+\alpha)(1+\beta)} \\
\hline
\phi & \frac{r^2\sigma_x\sigma_y}{2s(1+\alpha)(1+\beta)} & \frac{\sigma_y^2}{2s(1+\beta)} & \frac{r\sigma_y(1+\alpha-r^2)}{2s(1+\alpha)(1+\beta)} \\
\hline
t & \frac{r\sigma_x(1+\beta-r^2)}{2s(1+\alpha)(1+\beta)} & \frac{r\sigma_y(1+\alpha-r^2)}{2s(1+\alpha)(1+\beta)} & \frac{(1+\alpha)(1+\beta)-r^2\left(\frac{\alpha+\beta}{2}-\alpha\beta+2\right)+r^4}{s(1+\alpha)(1+\beta)} \\
\hline
\end{array}$$

The determinant of (37) is

$$(38)\qquad \frac{\sigma_x^2\sigma_y^2\left[(1-r^2)^3+(\alpha+\beta)(1-r^2)+\alpha\beta(1+r^2)\right]}{4s^3(1+\alpha)^2(1+\beta)^2}.$$

The variance of r_0 is given by the element in the lower right corner of (37). It can be readily shown that this variance is greater than $\frac{(1-r^2)^2}{s}$, the variance of the estimate of the correlation coefficient from ω_{xy} only—a rather surprising result.

The efficiency of θ, ϕ and t taken jointly is the ratio of the reciprocal of the determinant of (37) to B(m,n,s) in (13). That is,

$$(39)\qquad Eff(\theta,\phi,t) = \frac{(1-r^2)^3(1+\alpha)^2(1+\beta)^2}{\left[(1+\alpha)(1+\beta)-\alpha\beta r^4\right]\left[(1-r^2)^3+(\alpha+\beta)(1-r^2)+\alpha\beta(1+r^2)\right]},$$

which is less than unity except for the cases $r=0$ and $\alpha=\beta=0$.

6. Efficiency of the system θ, ϕ and $\frac{\xi}{\sqrt{\xi\eta}}$.

If we use $\frac{\xi}{\sqrt{\xi\eta}} = r_1$, say, which is the maximum likelihood estimate of r from ω_{xy}, instead of r_0 in section 5, and use the foregoing analysis of von Mises, we find the following matrix of variances and covariances for the asymptotic normal distribution of θ, ϕ and r_1 :

(40)

$$\begin{array}{c|c|c|c}
 & \theta & \phi & r_1 \\
\hline
\theta & \dfrac{\sigma_x^2}{2s(1+\alpha)} & \dfrac{r^2\sigma_x\sigma_y}{2s(1+\alpha)(1+\beta)} & \dfrac{r\sigma_x(1-r^2)}{2s(1+\alpha)} \\
\hline
\phi & \dfrac{r^2\sigma_x\sigma_y}{2s(1+\alpha)(1+\beta)} & \dfrac{\sigma_y^2}{2s(1+\beta)} & \dfrac{r\sigma_y(1-r^2)}{2s(1+\beta)} \\
\hline
r_1 & \dfrac{r\sigma_x(1-r^2)}{2s(1+\alpha)} & \dfrac{r\sigma_y(1-r^2)}{2s(1+\beta)} & \dfrac{(1-r^2)^2}{s}
\end{array}$$

The determinant of this matrix is

$$\frac{\sigma_x^2\sigma_y^2(1-r^2)^2\left[(1+\alpha)(1+\beta)-\frac{r^2}{2}(\alpha+\beta+2)\right]}{4s^2(1+\alpha)^2(1+\beta)^2}, \tag{41}$$

whose reciprocal provides us with the amount of information relative to σ_x, σ_y and r yielded by the estimates θ, ϕ and r_1. The efficiency of this system of estimates is given by the ratio of the reciprocal of (41) to (13), that is,

$$Eff(\theta,\phi,r_1)=\frac{(1-r^2)(1+\alpha)^2(1+\beta)^2}{\left[(1+\alpha)(1+\beta)-r^4\alpha\beta\right]\left[(1+\alpha)(1+\beta)-\frac{r^2}{2}(\alpha+\beta+2)\right]}.$$

By comparing the systems θ, ϕ, r_0 and θ, ϕ, r_1, we actually find the latter to be more efficient, since

$$\frac{(1-r^2)^2\left[(1+\alpha)(1+\beta)-\frac{r^2}{2}(\alpha+\beta+2)\right]}{\left[(1-r^2)^3+(\alpha+\beta)(1-r^2)+\alpha\beta(1+r^2)\right]}\leq 1,$$

which is the ratio of the reciprocal of (38) to that of (41). The equality holds ony when $r=0$ or $\alpha=\beta=0$.

The distribution $f(z,w,t)$ of $\bar{\xi}_0=z$, $\bar{\eta}_0=w$ and $r_1=t$ can be readily found from the distribution of ξ, η, ζ, u and v, which is included in (1), by making the following sets of transformations in succession,

$$\text{(a)}\quad \zeta=t\sqrt{\xi\eta}, \qquad d\zeta=\sqrt{\xi\eta}\,dt,$$

$$\text{(b)}\quad \begin{cases}\xi=\overline{1+\alpha}\,z-\alpha u & d\xi=\overline{1+\alpha}\,dz\\ \eta=\overline{1+\beta}\,z-\beta v & d\eta=\overline{1+\beta}\,dw\end{cases}$$

$$\text{(c)}\quad \begin{cases}u=\frac{1+\alpha}{\alpha}z(1-\theta) & du=-\frac{1+\alpha}{\alpha}z\,d\theta\\ v=\frac{1+\beta}{\beta}w(1-\phi) & dv=-\frac{1+\beta}{\beta}w\,d\phi.\end{cases}$$

The result can be expressed in closed form as the definite integral

$$(42)\quad f(z,w,t)=K(1-t^2)^{\frac{s-4}{2}}z^{\frac{s+m-4}{2}}w^{\frac{s+n-4}{2}}$$

$$\times\int_0^1\!\!\int_0^1 e^{-g}\theta^{\frac{s-3}{2}}(1-\theta)^{\frac{m-3}{2}}\phi^{\frac{s-3}{2}}(1-\phi^{\frac{n-3}{2}}\,d\theta\,d\phi.$$

where

$$K=\frac{(\frac{1}{2})^{\frac{N_1+N_2-4}{2}}N_1^{\frac{N_1-2}{2}}N_2^{\frac{N_2-2}{2}}\sigma_x^{-N_1+2}\sigma_y^{-N_2+2}(1-r^2)^{-\frac{s-1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{s-1}{2})\Gamma(\frac{s-2}{2})\Gamma(\frac{m-1}{2})\Gamma(\frac{n-1}{2})},$$

and

$$g = \frac{1}{2(1-r^2)}\left[\frac{N_1[1-r^2(1-\theta)]z}{\sigma_x^2} + \frac{N_2[1-r^2(1-\phi)]w}{\sigma_y^2} - \frac{2rt\sqrt{N_1N_2\theta\phi zw}}{\sigma_x\sigma_y}\right]$$

When $r=0$, (42) breaks into the product of three well known functions, two of which represent the distributions of the variances in samples having $s+m-2$ and $s+n-2$ degrees of freedom, and the third which is the distribution of the correlation coefficient in samples of s items from a normal population in which the correlation is zero.

IV. Summary.

Samples are considered from a bivariate normal population of x and y in which all of the members are not observed with respect to both x and y. Such a sample is broken into three parts ω_{xy}, ω_x and ω_y, where ω_{xy} is the set of s members observed with respect to both x and y, ω_x the set of m members observed with respect to x only and ω_y the remaining items observed with respect to y only.

Maximum likelihood estimates are found for the following sets of conditions:

(a) For given values of σ_x, σ_y and r, optimum estimates are found for the means a and b.

(b) For given values of a, b and r, optimum estimates are found for σ_x and σ_y.

(c) For given values of a and b, approximations are found for the optimum estimates of σ_x, σ_y and r.

Other sets of estimates considered are:

(1) Means a and b estimated independently from the x's and the y's respectively, of the sample ω.

(2) Maximum likelihood estimates of σ_x from ω_{xy} and ω_x and σ_y from ω_{xy} and ω_y, each estimated independently of the other. The estimate of $r\sigma_x\sigma_y$ is taken as the covariance from ω_{xy}. The characteristic function of these estimates is found.

(3) Estimates of σ_x and σ_y taken as the square root of the weighted averages of the variances from ω_{xy} and ω_x, and from ω_{xy} and ω_y respectively, with the estimate of r taken as the ratio of the covariance of ω_{xy} to the product of these estimates of the standard deviations.

(4) Estimates of σ_x and σ_y the same as in (3), with r estimated entirely from ω_{xy}.

The exact forms of the sampling distributions of the systems in (3) and (4) are found, as well as the asymptotic normal forms approached by these exact distributions as the size of the sample ω increases, subject to the condition that $\frac{m}{s}=\alpha$ and $\frac{n}{s}=\beta$ are constant. The limiting value of the variance of the estimate of r in (4) was found to be less than that of r in (3).

We have defined the amount of information available in a sample relative to any set of population parameters as the reciprocal of the determinant of the matrix of the limiting values, for large samples, of the variances and covariances of the maximum likelihood estimates of these parameters. It is shown that this determinant is smaller than that obtained from the asymptotic normal form approached by any other set of estimates of the same set of parameters. The amount of information relative to the parameters utilized by any other set of estimates is the reciprocal of the determinant of the matrix of the limiting values of the variances and covariances of this set of estimates. The measure of the efficiency of any set of estimates is taken as the ratio of the amount of information yielded by this set to the amount yielded by the maximum likelihood estimates. The efficiency thus defined was found for each of the sets of estimates (1), (3) and (4). It was found that the set (4) is more efficient than set (3).

S. S. Wilks

ON THE SAMPLING DISTRIBUTION OF THE MULTIPLE CORRELATION COEFFICIENT

By S. S. Wilks*

The problem of finding the distribution of the multiple correlation coefficient in samples from a normal population with a non-zero multiple correlation coefficient was solved in 1928 by Fisher[1] by the application of geometrical methods. In his derivation he used the facts that the population value ρ of the multiple correlation coefficient is invariant under linear transformations of the independent variates, and that the distribution of the multiple correlation coefficient is independent of all population parameters except ρ.

In this paper it will be shown that the distribution of the multiple correlation coefficient can be derived directly from Wishart's[2] generalized product moment distribution without making use of geometrical notions and the property of the invariance of ρ under linear transformations of the independent variates. Furthermore, it will not be necessary to show that the distribution will be independent of all population parameters except ρ

The population value of the multiple correlation coefficient between a variate x_1 and a set of variates $x_2, x_3, \cdots x_n$ is the ordinary correlation coefficient between x_1 and that linear function of the variates $x_2, x_3, \cdots x_n$ which will make this correlation a maximum. It can be expressed as $\rho^2 = 1 - \frac{\Delta}{\Delta_1}$ where Δ is the determinant of the correlations among all of the variates $x_1, x_2, \cdots x_n$ and Δ_1 is the determinant of correlations among the independent variates $x_2, x_3, \cdots x_n$ Denoting the sample value of ρ^2 by R^2 it is well known that $R^2 = 1 - \frac{D}{D_1}$, where D and D_1 are the determinants of sample correlations among the sets of variates $x_1, x_2, \cdots x_n$ and $x_2, x_3, \cdots x_n$ respectively.

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[1] R. A. Fisher, The general sampling distribution of the multiple correlation coefficient, Proceedings of the Royal Society of London, series A, vol. 121 (1928), pp. 654-73.

[2] John Wishart, The generalized product moment distribution in samples from a normal multivariate population, Biometrika, vol. 20A (1928) pp 32-52

Let us suppose a sample of N items to be drawn at random from the normal n-variate population whose distribution is

$$(1) \qquad \frac{\sqrt{A}}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}\sum_{i,j=1}^{n} A_{ij}(x_i - m_i)(x_j - m_j)}$$

where $A_{ij} = \frac{\Delta_{ij}}{\sigma_i \sigma_j \Delta}$, $\Delta = \left| \rho_{ij} \right|$ the determinant of correlations among the n variates, Δ_{ij} is the cofactor of ρ_{ij} in Δ, σ_i is the standard deviation of x_i and $A = \left| A_{ij} \right|$

In the sample, let

$$\bar{x}_i = \frac{1}{N} \sum_{\alpha=1}^{N} x_{i\alpha},$$

and

$$a_{ij} = \frac{1}{N} \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j),$$

where $x_{i\alpha}$ is the value of x_i for the α-th individual of the sample. Wishart[8] has proved that the simultaneous distribution function of the set $\{a_{ij}\}$, $(i, j = 1, 2, \cdots n)$ is

$$(2) \quad f(a) = \frac{\left(\frac{N}{2}\right)^{\frac{n(N-1)}{2}} A^{\frac{N-1}{2}}}{\pi^{\frac{n(n-1)}{4}} \Gamma\left(\frac{N-1}{2}\right) \Gamma\left(\frac{N-2}{2}\right) \cdots \Gamma\left(\frac{N-n}{2}\right)} e^{-\frac{N}{2}\sum_{i,j=1}^{n} A_{ij} a_{ij}} \left| a_{ij} \right|^{\frac{N-n-2}{2}}$$

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[8] J. Wishart, loc. cit.

where $|a_{ij}|$ is the determinant of the a's.

We shall define a moment-generating function $\phi(\alpha,k)$ as

$$(3) \qquad \phi(\alpha,k)=\int e^{\alpha a_{11}}|a_{ij}|^{k}|a_{pq}|^{-k}f(\bar{a})\,d\bar{a},$$

where the integration is to be taken over the field of all possible values of the a's and $|a_{pq}|$ is the cofactor of a_{11} in $|a_{ij}|$. From this definition of $\phi(\alpha,k)$, it is clear that $\left.\frac{\partial^h}{\partial\alpha^h}\phi(\alpha,k)\right|_{\alpha=0}$ is the product moment $E\left[a_{11}^{h+k}(1-R^2)^k\right]$. It will be shown that this expectation exists for $h=-k$ which will yield the k-th moment of $(1-R^2)$, from which the distribution of R^2 can be found.

To find $\phi(\alpha,k)$ we observe that since (2) is a probability function, its value over the field of all possible values of the a's is unity. Hence, we must have

$$(4) \qquad \int e^{-\frac{N}{2}\sum_{i,j=1}^{n}A_{ij}a_{ij}}|a_{ij}|^{\frac{N-n-2}{2}}d\bar{a}=G,$$

where $G=\dfrac{\pi^{\frac{n(n-1)}{4}}\Gamma(\frac{N-1}{2})\Gamma(\frac{N-2}{2})\cdots\Gamma(\frac{N-n}{2})}{(\frac{N}{2})^{\frac{n(N-1)}{2}}A^{\frac{N-1}{2}}}$. This relation holds for all positive values of $N>n$ and for all values of A_{ij} which will make the matrix $\|A_{ij}\|$ positive definite.

If $f(\bar{a})$ be integrated with respect to $a_{11}, a_{12}, \cdots a_{1n}$, the resulting form will clearly be the distribution of the set of a's contained in $|a_{pq}|$ and will be

$$(5) \quad \frac{\left(\frac{N}{2}\right)^{\frac{(n-1)(N-1)}{2}} B^{\frac{N-1}{2}}}{\pi^{\frac{(n-1)(n-2)}{4}}\, \Gamma\left(\frac{N-1}{2}\right)\cdots\Gamma\left(\frac{N-n+1}{2}\right)}\, e^{-\frac{N}{2}\sum_{p,q=2}^{n} B_{pq}a_{pq}}\, |a_{pq}|^{\frac{N-n-1}{2}},$$

where B_{pq} is the element in the p-th row and q-th column of the reciprocal form[4] of the determinant which is the co factor of the term in the first row and first column of the reciprocal form of $|A_{ij}|$. The value of B_{pq} in terms of correlation coefficients and standard deviations is $\frac{\Delta^{(1)}_{pq}}{\sigma_p \sigma_q \Delta^{(1)}}$, where $\Delta^{(1)} = \Delta_{11}$, and $\Delta^{(1)}_{pq}$ is the cofactor of ρ_{pq} in Δ_{11}. Furthermore, $B = |B_{pq}|$. Hence

$$(6) \quad \int e^{-\frac{N}{2}\sum_{i,j=1}^{n} A_{ij}a_{ij}}\, |a_{ij}|^{\frac{N-n-2}{2}}\, da_1\, d[a-a_1]$$

$$= \pi^{\frac{n-1}{2}} \left(\frac{N}{2}\right)^{-\frac{N-1}{2}} \left(\frac{B}{A}\right)^{\frac{N-1}{2}} \Gamma\left(\frac{N-n}{2}\right) e^{-\frac{N}{2}\sum_{p,q=2}^{n} B_{pq}a_{pq}}\, |a_{pq}|^{\frac{N-n-1}{2}}\, d[a-a_1],$$

where $da_1 = da_{11}da_{12}\cdots da_{1n}$ and $d[a-a_1]$ is the product of the differentials of all a's in $|a_{pq}|$ $(p,q = 2,3,\cdots n)$.

Now, it is clear that (6) is an identity for all values of N and the population parameters σ_i and ρ_{ij} $(i,j = 1,2,\cdots m;\ i \neq j)$, for which both sides of (6) exist. Thus, we can perform the following operations on (6):

(a). Replace N by $N+2k$.

(b). Replace σ_i by $\sigma_i\sqrt{\frac{N+2k}{N}}$, $(i = 1,2,\cdots n)$

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[4] By the reciprocal form of a determinant $|c_{ij}|$ we mean the determinant formed by replacing each element c_{ij} by the ratio $\frac{C_{ij}}{C}$ where C_{ij} is the cofactor of c_{ij} and $C = |c_{ij}|$.

(c). Replace A_{11} by $A_{11} - \frac{2\alpha}{N}$.

(d). Multiply both sides of the identity by $\frac{1}{G_0}$.

(e). Multiply both sides by $|a_{pq}|^{-k}$.

Accordingly, we find that the integral of the left side of (6) over all possible values of the a's is the definition of $\phi(\alpha,k)$, which must be equal to the integral of the right side over the field of all possible values of the a's in $|a_{pq}|$ But the value of the integral of the right side can be deduced at once from (4). Hence, we finally obtain,

$$(7) \qquad \phi(\alpha,k) = \left(\frac{N}{2}\right)^{-k} A^{\frac{N-1}{2}} A_\alpha^{-\frac{N-1}{2}-k} B_\alpha^{k} \frac{\Gamma\left(\frac{N-n}{2}+k\right)}{\Gamma\left(\frac{N-n}{2}\right)},$$

where A_α is the determinant A with A_{11} replaced by $A_{11} - \frac{2\alpha}{N}$, and B_α is the reciprocal of the cofactor of the element in the first row and first column of the reciprocal form of A_α.
That is,

$$(8) \qquad B_\alpha = \frac{A_\alpha^{n-1}}{|\bar{A}_{\alpha pq}|},$$

where $\bar{A}_{\alpha pq}$ is the cofactor of the element in the p-th row and q-th column of A_α, $(p,q = 2,3,\cdots n)$ The value of $|\bar{A}_{\alpha pq}|$ can be readily found by writing

$$|\bar{A}_{\alpha pq}| = \frac{|\bar{A}_{\alpha pq}|\,|\bar{A}_{\alpha ij}|}{A_\alpha},$$

where $A_{\alpha ij} \equiv A_{ij}$ except for $i=j=1$ and $A_{\alpha 11} = A_{11} - \frac{2\alpha}{N}$. Increasing $|\bar{A}_{\alpha pq}|$ to an n th order determinant by inserting, as first row and first column, an additional row and column which will not change the value of the determinant, and multiplying it by $|A_{\alpha 1q}|$ we find

$$|\bar{A}_{\alpha pq}| = A_\alpha^{n-2}\left(A_{11} - \frac{2\alpha}{N}\right).$$

Therefore,

$$B_\alpha = \frac{A_\alpha}{(A_{11} - \frac{2\alpha}{N})}\,.$$

Substituting this for B_α in (7) and using the fact that

$A_\alpha = A - \frac{2\alpha}{N}\bar{A}_{11}$, we finally obtain

$$(9)\quad \phi(\alpha,k) = \left(\frac{NA}{2\bar{A}_{11}}\right)^{\frac{N-1}{2}} \left(\frac{NA}{2\bar{A}_{11}} - \alpha\right)^{-\frac{N-1}{2}} \left(\frac{NA_{11}}{2} - \alpha\right)^{-k} \frac{\Gamma\left(\frac{N-n}{2}+k\right)}{\Gamma\left(\frac{N-n}{2}\right)}.$$

Thus, it is evident that $\phi(\alpha,k)$ exists for sufficiently small values of α Let us write

$$\left(\frac{NA}{2\bar{A}_{11}} - \alpha\right)^{-\frac{N-1}{2}} = \left(\frac{NA_{11}}{2} - \alpha\right)^{-\frac{N-1}{2}} \left[1 - \frac{\frac{N}{2}\left(A_{11} - \frac{A}{\bar{A}_{11}}\right)}{\left(\frac{NA_{11}}{2} - \alpha\right)}\right]^{-\frac{N-1}{2}},$$

and expand the second factor on the right into a Taylor series. Substituting in (9), we have the convergent series

$$(10)\quad \begin{aligned} \phi(\alpha,k) &= \left(\frac{NA}{2\bar{A}_{11}}\right)^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{N-n}{2}+k\right)}{\Gamma\left(\frac{N-n}{2}\right)} \\ &\times \sum_{i=0}^{\infty} \frac{\left(\frac{NA_{11}}{2} - \alpha\right)^{-k-\frac{N-1}{2}-i} \left(\frac{N}{2}\right)^{i} \left(A_{11} - \frac{A}{\bar{A}_{11}}\right)^{i} \Gamma\left(\frac{N-1}{2}+i\right)}{i!\,\Gamma\left(\frac{N-1}{2}\right)}. \end{aligned}$$

For the coefficient of $\frac{\alpha^h}{h!}$ in the expansion of the right side of (10) in powers of α, we find

$$(11)\quad \begin{aligned} &\left(\frac{NA_{11}}{2}\right)^{-k-h} \left(\frac{A}{\bar{A}_{11}A_{11}}\right)^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{N-n}{2}+k\right)}{\Gamma\left(\frac{N-n}{2}\right)} \\ &\times \sum_{i=0}^{\infty} \frac{\left(1 - \frac{A}{A_{11}\bar{A}_{11}}\right)^{i} \Gamma\left(\frac{N-1}{2}+i\right) \Gamma\left(\frac{N-1}{2}+k+h+i\right)}{i!\,\Gamma\left(\frac{N-1}{2}\right) \Gamma\left(\frac{N-1}{2}+k+i\right)} \end{aligned}$$

which is the definition of $E\left[a_{11}^{h+k}(1-R^2)^k\right]$. We observe that (11) exists for all values of k and h for which $\frac{N-n}{2}+k>0$, and $\frac{N-1}{2}+h+k>0$. Placing $h=-k$ and pointing out that $\frac{A}{A_{11}\bar{A}_{11}}=1-\rho^2$, we have as the k-th moment of $1-R^2$,

$$M_k\left[(1-R^2)\right]=E\left[(1-R^2)^k\right]=$$

(12)
$$\frac{(1-\rho^2)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-n}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}\sum_{i=0}^{\infty}\frac{\rho^{2i}\,\Gamma^2\left(\frac{N-1}{2}+i\right)\Gamma\left(\frac{N-n}{2}+k\right)}{i!\,\Gamma\left(\frac{N-1}{2}+k+i\right)}.$$

By using the relation

$$\frac{\Gamma\left(\frac{N-n}{2}+k\right)}{\Gamma\left(\frac{N-1}{2}+k+i\right)}=\frac{1}{\Gamma\left(\frac{n-1}{2}+i\right)}\int_0^1(1-\theta)^{\frac{N-n}{2}+k-1}\theta^{\frac{n-1}{2}+i-1}\,d\theta$$

we can write (12) in the form

(13)
$$E\left[(1-R^2)^k\right]=\frac{(1-\rho^2)^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-n}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}\times\sum_{i=0}^{\infty}\int_0^1\frac{\rho^{2i}(1-\theta)^{\frac{N-n}{2}+k-1}\theta^{\frac{n-1}{2}+i-1}\,\Gamma^2\left(\frac{N-1}{2}+i\right)}{i!\,\Gamma\left(\frac{n-1}{2}+i\right)}\,d\theta.$$

The series in (13) is uniformly convergent in θ for $0\leq\theta\leq 1$ and therefore, we can interchange the order of summation and integration and write

(14)
$$E\left[(1-R^2)^k\right]=\int_0^1(1-\theta)^k\phi(\theta)\,d\theta,$$

where

$$\phi(\theta) = \frac{(1-\rho^2)^{\frac{N-1}{2}}(1-\theta)^{\frac{N-n}{2}-1}\,\theta^{\frac{n-1}{2}-1}}{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-n}{2}\right)}$$

$$\times \sum_{i=0}^{\infty} \frac{\rho^{2i}\theta^{i}\,\Gamma^{2}\left(\frac{N-1}{2}+i\right)}{i!\,\Gamma\left(\frac{n-1}{2}+i\right)}. \tag{15}$$

Thus, we have a distribution function of a variable θ such that the k-th moment of θ is identical with the k-th moment of R^2 for all positive values of k. It follows from Stekloff's[5] theory of closure that $\phi(\theta)$ must be the only continuous solution of (14), where $E\left[(1-R^2)^k\right]$ is defined as (12). Therefore, the distribution of R^2 is identical with that of θ and can be written finally as

$$df = \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}$$

$$\times (1-\rho^2)^{\frac{N-1}{2}}(1-R^2)^{\frac{N-n}{2}-1}(R^2)^{\frac{n-3}{2}}F\left[\tfrac{N-1}{2},\tfrac{N-1}{2},\tfrac{n-1}{2},\rho^2R^2\right]d(R^2). \tag{16}$$

which is the distribution found by Fisher except that he uses the notation $n_1 = n-1$, the number of independent variates, and $n_1+n_2+1 = N$, the sample number.

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[5] W. Stekloff: Quelques applications nouvelles de la thèorie de fermèture au probleme de representation approchée des functiones et au probleme des moments, Memoire de l'Academie Imperial des Sciences de St. Petersburg, vol. 32, no. 4, (1914).

S. S. Wilks

CURVE APPROXIMATION BY MEANS OF FUNCTIONS ANALOGOUS TO THE HERMITE POLYNOMIALS.

By HERRICK E. H. GREENLEAF

I. Introduction

In an article by J. P. Gram entitled "Ueber die Entwickelung reeler Functionem in Reihen mittelst der Methode der kleinsten Quadrate"[1] a unique procedure is set forth which leads to a very great simplification in the usual method of curve fitting by the method of least squares. That this method has not been given more consideration is probably due to lack of knowledge of its existence, rather than to lack of appreciation of its merit. Edward Condon,[2] Raymond T. Birge and John D. Shea[3] developed formulas by means of which curves can be fitted to certain types of data. Later, Harold T. Davis and Voris V Latshaw[4] developed specific formulas, with tables of coefficients, by means of which curves of the second to the seventh degree can be fitted to the data with a minimum amount of computation. In a later paper, Professor Davis[5] has employed Gram's method and in this way has developed a set of functions analagous to the Legendre polynominals.

The purposes of the present paper are:

(1) To develop formulas for fitting curves of the second to the sixth degree to given data by the method of least squares where the $n+1$ frequencies of the data have the terms of the expansion of $(\frac{1}{2}+\frac{1}{2})^n$ as weighting factors.

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[1] Journal fur Mathematik, Vol. 94, 1894, pp. 41-73, especially pp. 42-46.

[2] "The Rapid Fitting of a Certain Class of Empirical Formulae by the Method of Least Squares." Univ. of California Pub. in Math. Vol. 2, No. 4, pp. 55-66, March 1927.

[3] "A Rapid Method for Calculating the Least Squares Solution of a Polynomial of any Degree." University of California Publications in Mathematics. Vol. 2, No. 5, pp. 67-118, March 1927.

[4] "Formulas for the Fitting of Polynomials to Data by the method of Least Squares." Annals of Mathematics, Second Series, Vol. 31, No. 1, Jan. 1930, pp. 52-78.

(2) To develop by Gram's method a set of functions analogous to the Hermite polynomials, by means of which curves of the second to the eighth degree can be fitted to data under the same conditions as in (1).

(3) To study the properties of these functions, finding a generating function, a recurrence formula, a second order difference equation, and giving other methods for deriving them.

(4) To apply the functions in finding a curve to fit given data

(5) To furnish tables to facilitate rapid calculation of the coefficients of the required equation.

II. Development By Ordinary Method of Least Squares

Suppose we have given data in which the variates, x , are equally spaced, the observations being weighted with the binomial coefficients, having the origin at the mean. Thus, let the given data be

x	x_{-p}	x_{-p+1}	x_{-p+2}	$\cdot\ \cdot$	x_{-1}	x_0	x_1	$\cdot\ \cdot\ \ \cdot\ \cdot$	x_p
y_x	y_{-p}	y_{-p+1}	y_{-p+2}	$\cdot\ \cdot\ \cdot$	y_{-1}	y_0	y_1	$\cdot$	y_p
w_x	C_{-p}	C_{-p+1}	C_{-p+2}	$\cdot\ \ \cdot\ \cdot$	C_{-1}	C_0	C_1	$\cdot\ \cdot$	C_p

where $x_{i+1} - x_i$ is constant,

and $$C_x = \frac{(2p)!}{(p+x)!(p-x)!}\frac{1}{2}^{p+x}\frac{1}{2}^{p\ x}$$

Since the x differences are constant, it is possible, without the loss of generality, to replace the x's by their corresponding subscripts. It is evident that the problem as set forth deals with

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[5] "Polynomial Approximation by the Method of Least Squares." Annals of Mathematics.

data having an odd number of classmarks. If there should be an even number, the data can be modified by leaving out one of the end variates, or by adding an item by extrapolation. It is also possible to transform these functions by moving the origin to the extreme left, thus having x vary from o to n. If this is done, the functions in part 3 of this paper will reduce to those generated by equation (18) in Gram's article.

It is required to find the coefficients $a_{i,k}$ in the equation

$$(1) \quad y = y'C_x = \left[a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \cdots\cdots a_{n,n}x^n\right]C_x$$

such that

$$\sum_{-p}^{p} C_x(y_x - y')^2 = \sum_{-p}^{p} C_x\left[y_x - a_{n,0} - a_{n,1}x - a_{n,2}x^2 \cdots - a_{n,n}x^n\right]^2$$

shall be a minimum.

In this section, the ordinary method of least squares is used, leading to the $n+1$ equations

$$(2) \quad \begin{aligned} M_0 &= a_{n,0}m_0 + a_{n,2}m_2 + a_{n,4}m_4 + \cdots\cdots \\ M_1 &= a_{n,1}m_2 + a_{n,3}m_4 + a_{n,5}m_6 + \cdots \\ M_2 &= a_{n,0}m_2 + a_{n,2}m_4 + a_{n,4}m_6 + \cdots \\ M_3 &= a_{n,1}m_4 + a_{n,3}m_6 + a_{n,5}m_8 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where

$$(3) \quad M_r = \sum_{x=-p}^{p} y_x x^r \quad \text{and} \quad m_r = \sum_{x=-p}^{p} C_x x^r$$

(It is evident from the symmetry of the distribution that all moments, m_r, with r odd will be identically zero.)

These $n+1$ equations can be solved more readily by dividing them into two sets, one containing the coefficients of subscripts $a_{n,2r}$, the other set containing the subscripts $a_{n,2r+1}$. Thus we get

$$M_0 = a_{n,0}m_0 + a_{n,2}m_2 + a_{n,4}m_4 + \cdots\cdots;$$

$$(4) \quad M_2 = a_{n,0}m_2 + a_{n,2}m_4 + a_{n,4}m_6 + \cdots\cdots$$

$$M_4 = a_{n,0}m_4 + a_{n,2}m_6 + a_{n,4}m_8 +$$

$$\cdots\cdots\cdots\cdots\cdots$$

and

$$M_1 = a_{n,1}m_2 + a_{n,3}m_4 + a_{n,5}m_6 + \cdots$$

$$(4a) \quad M_3 = a_{n,1}m_4 + a_{n,3}m_6 + a_{n,5}m_8 + \cdots$$

$$M_5 = a_{n,1}m_6 + a_{n,3}m_8 + a_{n,5}m_{10} + \cdots$$

$$\cdots\cdots\cdots\cdots\cdots$$

The computation of the moments, m_r, may be accomplished in the following manner:

$$(5) \qquad C_x = \frac{(2p)!}{(p+x)!(p-x)!}\,\frac{1}{2^{2p}}$$ is the

general term in the expansion of $(\frac{1}{2}+\frac{1}{2})^{2p}$,

$$\frac{C_{x+1}}{C_x} = \frac{(2p)!}{(p+x+1)!(p-x-1)!}\,\frac{1}{2^{2p}}\cdot\frac{(p+x)!(p-x)!}{(2p)!}\,2^{2p} = \frac{p-x}{p+x+1}$$

$$(6) \qquad \therefore\; C_{x+1}(p+x+1) = (p-x)C_x.$$

Multiplying each side of (6) by $(x+1)^k$ and summing from

$x = -p$ to $x = p$, we get

$$\sum_{-p}^{p} C_{x+1}(p+x+1)(x+1)^k = \sum_{-p}^{p} (p-x)(x+1)^k C_x \qquad \text{or}$$

$$\sum_{-p}^{p} C_{x+1}(p)(x+1)^k + \sum_{-p}^{p} C_{x+1}(x+1)^{k+1} = \sum_{-p}^{p} p(x+1)^k C_x - \sum_{-p}^{p} x(x+1)^k C_x$$

$$= p\left[\sum_{-p}^{p} C_x x^k + \binom{k}{1}\sum_{-p}^{p} C_x x^{k-1} + \binom{k}{2}\sum_{-p}^{p} C_x x^{k-2} + \qquad \cdot\right]$$

$$- \sum_{-p}^{p}\left[C_x x^{k+1} + \binom{k}{1} C_x x^k + \binom{k}{2} C_x x^{k-1} + \quad \cdots \right].$$

By virtue of (3), this becomes

$$p m_k + m_{k+1} = p\left[m_k + \binom{k}{1} m_{k-1} + \binom{k}{2} m_{k-2} + \cdots \right]$$

$$(7) \qquad - \left[m_{k+1} + \binom{k}{1} m_k \binom{k}{2} m_{k-1} + \cdots \right].$$

Combining the terms in m_i, and recalling that all odd moments are identically zero, we reduce equation (7) to

$$(8) \quad 2m_{k+1} = \left[\binom{k}{1}p - \binom{k}{2}\right] m_{k-1} + \left[\binom{k}{3}p - \binom{k}{4}\right] m_{k-3} + \left[\binom{k}{5}p - \binom{k}{6}\right] m_{k-5} + \cdots$$

By means of this recurrence relationship, the moments are found to be

$$m_0 = 1$$

$$m_2 = \frac{p}{2}$$

$$m_4 = \frac{3p^{(2)}}{4} + \frac{p}{2}$$

$$m_6 = \frac{15p^{(3)}}{8} + \frac{15p^{(2)}}{4} + \frac{p}{2}$$

$$m_8 = \frac{105p^{(4)}}{16} + \frac{105p^{(3)}}{4} + \frac{63p^{(2)}}{4} + \frac{p}{2}$$

$$m_{10} = \frac{945p^{(5)}}{32} + \frac{1575p^{(4)}}{8} + \frac{2205p^{(3)}}{8} + \frac{255p^{(2)}}{4} + \frac{p}{2}$$

$$m_{12} = \frac{10395p^{(6)}}{64} + \frac{51975p^{(5)}}{32} + \frac{65835p^{(4)}}{16} + 2640p^{(3)} + \frac{1023p^{(2)}}{4} + \frac{p}{2}$$

$$m_{14} = \frac{135135p^{(7)}}{128} + \frac{945945p^{(6)}}{64} + \frac{945945p^{(5)}}{16} + 75075p^{(4)} + \frac{195195p^{(3)}}{8} + \frac{4095p^{(2)}}{4} + \frac{p}{2}$$

where $p^{(n)} = p(p-1)(p-2)\cdots(p-n+1)$.

When expanded, the values are

$$m_0 = 1$$

$$m_2 = \frac{p}{2}$$

$$m_4 = \frac{p(3p-1)}{4}$$

$$m_6 = \frac{p(15p^2-15p+4)}{8}$$

$$m_8 = \frac{p(105p^3-210p^2+147p-34)}{16}$$

$$m_{10} = \frac{p(945p^4-3150p^3+4095p^2-2370p+496)}{32}$$

$$m_{12} = \frac{p(10395p^5-51975p^4+107415p^3-111705p^2}{64} + \frac{56958p-11056)}{64}$$

$$m_{14} = \frac{p(135135p^6 - 945945p^5 + 2837835p^4 - 4579575p^3}{128}$$

$$+ \frac{4114110p^2 - 1911000p + 349504)}{128}$$

Knowing the moments, it is now possible, by explicit calculation, to find the coefficients, $a_{n,i}$, in (1) for special cases.

Case I. Linear

$$y = (a_{1,0} + a_{1,1}x)\,C_x .$$

(4) and (4a) both reduce to single equations

$$(9)\quad (a)\quad \begin{aligned} M_0 &= m_0 a_{1,0}, \\ a_{0,1} &= \frac{M_0}{m_0} = M_0, \end{aligned} \qquad (b)\quad \begin{aligned} M_1 &= m_2 a_{1,1}, \\ a_{1,1} &= A_1 M_1, \end{aligned}$$

where $A_1 = \frac{2}{p}$.

Case II. Quadratic.

$$y = (a_{2,0} + a_{2,1}x + a_{2,2}x^2)\,C_x .$$

This leads to the two sets of equations,

$$(a)\quad \begin{aligned} M_0 &= m_0 a_{2,0} + m_2 a_{2,2}, \\ M_2 &= m_2 a_{2,0} + m_4 a_{2,2}, \end{aligned}$$

and

$$(b)\quad M_1 = m_2 a_{2,1} .$$

(Notice that one set of equations in each case is identical with a set in the preceding case. Therefore, only one of the two sets needs to be solved.)

(b) is identical with (b) in (9),

$$a_{2,1} = a_{1,1} = A_1 M_1 .$$

Solving (a), we have

(10) $$a_{2,0} = A_2 M_0 + B_2 M_2 ,$$
$$a_{2,2} = B_2 M_0 + C_2 M_2 ,$$

where $A_2 = \frac{3p-1}{2p-1}$, $B_2 = -\frac{2}{2p-1}$,

$$C_2 = \frac{4}{p(2p-1)} .$$

Case III. Cubic.

$$y = (a_{3,0} + a_{3,1} x + a_{3,2} x^2 + a_{3,3} x^3) Cx$$

The resulting equations are

(a) $$M_0 = m_0 a_{3,0} + m_2 a_{3,2} ,$$
$$M_2 = m_2 a_{3,0} + m_4 a_{3,2} ,$$

(b) $$M_1 = m_2 a_{3,1} + m_4 a_{3,3} ,$$
$$M_3 = m_4 a_{3,1} + m_6 a_{3,3} ,$$

$$(11)\quad \begin{aligned} a_{3,0} &= a_{2,0} = A_2 M_0 + B_2 M_2, \\ a_{3,2} &= a_{2,2} = B_2 M_0 + C_2 M_2, \\ a_{3,1} &= A_3 M_1 + B_3 M_3, \\ a_{3,3} &= B_3 M_1 + C_3 M_3, \end{aligned}$$

where $$A_3 = \frac{2(15p^2-15p+4)}{d_3},$$

$$B_3 = \frac{-4(3p-1)}{d_3},$$

$$C_3 = \frac{8}{d_3},$$

$$d_3 = 3p(p-1)(2p-1).$$

Case IV. Quartic.

$$y = (a_{4,0} + a_{4,1}x + a_{4,2}x^2 + a_{4,3}x^3 + a_{4,4}x^4)C_x.$$

The coefficients with the second subscript odd, are identical with those in case III. The other coefficients are found from the equations

$$M_0 = m_0 a_{4,0} + m_2 a_{4,2} + m_4 a_{4,4},$$

$$M_2 = m_2 a_{4,0} + m_4 a_{4,2} + m_6 a_{4,4},$$

$$M_4 = m_4 a_{4,0} + m_6 a_{4,2} + m_8 a_{4,4}.$$

$$\therefore a_{4,1} = a_{3,1},$$

$$a_{4,3} = a_{3,3}.$$

and

$$a_{4,0} = A_4 M_0 + B_4 M_2 + C_4 M_4,$$

$$(12)\qquad \begin{aligned} a_{4,2} &= B_4 M_0 + D_4 M_2 + E_4 M_4, \\ a_{4,4} &= C_4 M_0 + E_4 M_2 + F_4 M_4, \end{aligned}$$

where

$$A_4 = \frac{(15p^2-25p+6)}{2d'_4},$$

$$B_4 = \frac{-10(p-1)}{d'_4},$$

$$C_4 = \frac{2}{d'_4},$$

$$d'_4 = (2p-1)(2p-3).$$

$$D_4 = \frac{4(24p^2-39p+17)}{d_4},$$

$$E_4 = -\frac{8(3p-2)}{d_4},$$

$$F_4 = \frac{8}{d_4},$$

$$d_4 = 3p(p-1)(2p-1)(2p-3).$$

Case V. Quintic.

$$y = (a_{5,0} + a_{5,1}x + a_{5,2}x^2 + a_{5,3}x^3 + a_{5,4}x^4 + a_{5,5}x^5)C_x.$$

As in the previous case, we have at once

$$a_{5,0} = a_{4,0},$$

$$a_{5,2} = a_{4,2},$$

$$a_{5,4} = a_{4,4},$$

and

$$M_1 = m_2 a_{5,1} + m_4 a_{5,3} + m_6 a_{5,5},$$

$$M_3 = m_4 a_{5,1} + m_6 a_{5,3} + m_8 a_{5,5},$$

$$M_5 = m_6 a_{5,1} + m_8 a_{5,3} + m_{10} a_{5,5},$$

from which

$$a_{5,1} = A_5 M_1 + B_5 M_3 + C_5 M_5,$$

$$(13) \qquad a_{5,3} = B_5 M_1 + D_5 M_3 + E_5 M_5,$$

$$a_{5,5} = C_5 M_1 + E_5 M_3 + F_5 M_5,$$

where

$$A_5 = \frac{(525p^4 - 2100p^3 + 2835p^2 - 1480p + 276)}{d_5},$$

$$B_5 = \frac{-(20)(21p^3 - 63p^2 + 56p - 12)}{d_5},$$

$$C_5 = \frac{4(15p^2 - 25p + 6)}{d_5},$$

$$D_5 = \frac{40(12p^2 - 27p + 16)}{d_5},$$

$$E_5 = \frac{-80(p-1)}{d_5},$$

$$F_5 = \frac{16}{d_5},$$

$$d_5 = 15p(p-1)(p-2)(2p-1)(2p-3).$$

Case VI. Sextic.

$$y = (a_{6,0} + a_{6,1}x + a_{6,2}x^2 + a_{6,3}x^3 + a_{6,4}x^4 + a_{6,5}x^5 + a_{6,6}x^6)C_x$$

As before, we have

$$a_{6,1} = a_{5,1},$$

$$a_{6,3} = a_{5,3},$$

$$a_{6,5} = a_{5,5}.$$

From the equations

$$M_0 = m_0 a_{6,0} + m_2 a_{6,2} + m_4 a_{6,4} + m_6 a_{6,6},$$

$$M_2 = m_2 a_{6,0} + m_4 a_{6,2} + m_6 a_{6,4} + m_8 a_{6,6},$$

$$M_4 = m_4 a_{6,0} + m_6 a_{6,2} + m_8 a_{6,4} + m_{10} a_{6,6},$$

$$M_6 = m_6 a_{6,0} + m_8 a_{6,2} + m_{10} a_{6,4} + m_{12} a_{6,6},$$

we obtain

$$\begin{aligned} a_{6,0} &= A_6 M_0 + B_6 M_2 + C_6 M_4 + D_6 M_6, \\ a_{6,2} &= B_6 M_0 + E_6 M_2 + F_6 M_4 + G_6 M_6, \\ a_{6,4} &= C_6 M_0 + F_6 M_2 + H_6 M_4 + I_6 M_6, \\ a_{6,6} &= D_6 M_0 + G_6 M_2 + I_6 M_4 + J_6 M_6, \end{aligned} \qquad (14)$$

where

$$A_6 = \frac{3(35p^3 - 140p^2 + 147p - 30)}{2d'_6},$$

$$B_6 = -\frac{7(15p^2 - 48p + 28)}{d'_6},$$

$$C_6 = \frac{14(3p-5)}{d'_6},$$

$$D_6 = -\frac{4}{d'_6},$$

$$d'_6 = 3(2p-1)(2p-3)(2p-5).$$

$$E_6 = \frac{2(3465p^4 - 18270p^3 + 33915p^2 - 25950p + 7216)}{d_6}$$

$$F_6 = -\frac{20(171p^3 - 681p^2 + 846p - 304)}{d_6},$$

$$G_6 = \frac{8(45p^2-105p+46)}{d_6},$$

$$H_6 = \frac{40(51p^2-147p+110)}{d_6},$$

$$I_6 = -\frac{80(3p-4)}{d_6},$$

$$J_6 = \frac{32}{d_6},$$

$$d_6 = 45p(p-1)(p-2)(2p-1)(2p-3)(2p-5).$$

Tables for all the coefficients, A_1, A_2, B_2 · · · to J_6 , to ten significant figures for p from 1 to 20 will be found at the end of this paper

Special attention is directed to the last coefficient in each case, $a_{r,r}$, as reference will be made to it later. It is desirable to be able to compute this coefficient without having to solve a set of equations.

Let $P_{(n)}$ represent the determinant

$$\text{(15a)} \qquad \begin{vmatrix} m_0 & m_2 & m_4 & \cdots & m_n \\ m_2 & m_4 & m_6 & \cdots & m_{n+2} \\ m_4 & m_6 & m_8 & \cdots & m_{n+4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ m_n & m_{n+2} & m_{n+4} & \cdots & m_{2n} \end{vmatrix}$$

for n an even integar, and

$$
(15b)\qquad \begin{vmatrix}
m_2 & m_4 & m_6 & \cdot & \cdot & \cdot & \cdot & m_{n+1} \\
m_4 & m_6 & m_8 & \cdot & \cdot & \cdot & \cdot & m_{n+3} \\
m_0 & m_8 & m_{10} & \cdot & \cdot & \cdot & \cdot & m_{n+5} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
m_{n+1} & m_{n+3} & m_{n+5} & \cdot & & & & m_{2n}
\end{vmatrix}
$$

for n odd. By definition

$$P_{(-2)} = P_{(-1)} = P_{(0)} = 1.$$

Let $P_{(n,M)}$ denote the same determinants, the last column being replaced by $M_0, M_2, M_4 \cdot \cdot \cdot M_n$, or by $M_1, M_3, M_5 \cdot \cdot M_n$ according to whether n is even or odd. Similarly, $P_{(n,x)}$ will represent the original determinants, the last column being replaced by $1, x^2, x^4, \cdot \cdot x^n$ or by $x, x^3, x^5, \cdot \cdot \cdot \cdot \cdot \cdot \cdot x^n$ for n even or odd respectively. It is clear from the normal equations in case r, that

$$
(16)\qquad a_{r,r} = \frac{P_{(r,M)}}{P_{(r)}} = \frac{\begin{vmatrix}
m_0 & m_2 & \cdot & \cdot & \cdot & \cdot & \cdot & M_0 \\
m_2 & m_4 & \cdot & \cdot & \cdot & \cdot & \cdot & M_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
m_r & m_{r+2} & \cdot & \cdot & \cdot & \cdot & \cdot & M_r
\end{vmatrix}}{\begin{vmatrix}
m_0 & m_2 & \cdot & \cdot & \cdot & \cdot & \cdot & m_r \\
m_2 & m_4 & \cdot & \cdot & \cdot & \cdot & \cdot & m_{r+2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
m_r & m_{r+2} & \cdot & \cdot & & & \cdot & m_{2r}
\end{vmatrix}} \quad \text{for } n \text{ even.}
$$

Thus, for $r = 4$

we have,
$$a_{4,4} = \frac{\begin{vmatrix} m_0 & m_2 & M_0 \\ m_2 & m_4 & M_2 \\ m_4 & m_6 & M_4 \end{vmatrix}}{\begin{vmatrix} m_0 & m_2 & m_4 \\ m_2 & m_4 & m_6 \\ m_4 & m_6 & m_8 \end{vmatrix}}$$

The case with which the determinants, $P_{(n)}$, can be evaluated is not at once evident. It will be shown later that

(17) $$P_{(n)} = \frac{n!(2p)^{(n)}}{2^{2n}} P_{(n-2)} \qquad \text{(See (47)} \; .$$

(18) where $(2p)^{(n)} = (2p)(2p-1)(2p-2)\cdots(2p-n+1)$.

Starting with $P_{(0)} = 1$, we have

$$P_{(2)} = \frac{2!(2p)^{(2)}}{2^{2.2}} \cdot 1 \, ,$$

$$P_{(4)} = \frac{4!(2p)^{(4)}}{2^{2.4}} \cdot \frac{2!(2p)^{(2)}}{2^{2\,2}} = \frac{4!2!(2p)^{(4)}(2p)^{(2)}}{2^{2(2+4)}} \, ,$$

$$P_{(6)} = \frac{6!\,4!\,2!\,(2p)^{(6)}(2p)^{(4)}(2p)^{(2)}}{2^{2(6+4+2)}}.$$

Similarly, since $P_{(1)} = \frac{2p}{2^2}$,

$$P_{(3)} = \frac{3!\,1!\,(2p)^{(3)}(2p)}{2^{2(3+1)}}.$$

It is clear that for n even, or odd,

$$(19)\ P_{(n)} \frac{n!(n-2)!(n-4)! \cdot\cdot\cdot\cdot\cdot (2p)^{(n)}(2p)^{(n-2)}(2p)^{(n-4)}\cdots}{2^{2(n+\overline{n-2}+\overline{n-4}.\ .\ .\ .\ .)}},$$

each series ending with $n-r=0$ or 1, for n even, or odd,

With the recurrence formula (17) or the general formula (19) it is a simple matter to evaluate $P_{(n)}$ a list of which follows:

$$P_{(0)} = 1,$$

$$P_{(1)} = \frac{p}{2},$$

$$P_{(2)} = \frac{p(2p-1)}{4},$$

$$P_{(3)} = \frac{3p^2(p-1)(2p-1)}{16},$$

$$P_{(4)} = \frac{3p^2(p-1)(2p-1)^2(2p-3)}{32},$$

$$
\begin{aligned}
P_{(5)} &= \frac{45p^3(p-1)^2(p-2)(2p-1)^2(2p-3)}{256}, \\
P_{(6)} &= \frac{135p^3(p-1)^2(p-2)(2p-1)^3(2p-3)^2(2p-5)}{1024}, \\
P_{(7)} &= \frac{14{,}175\,p^4(p-1)^3(p-2)^2(p-3)(2p-1)^3(2p-3)^2(2p-5)}{16{,}384}, \\
P_{(8)} &= \frac{42{,}525\,p^4(p-1)^3(p-2)^2(p-3)(2p-1)^4}{32{,}768} \\
&\qquad \frac{(2p-3)^3(2p-5)^2(2p-7)}{32{,}768}.
\end{aligned}
\tag{20}
$$

III. Development By Gram's Method.

Let the variates, x_i, the observations, y_x, and the weights, C_x, be given as before. We assume that there exists a set of functions

$$\phi_0(x), \qquad \phi_1(x), \qquad \phi_2(x) \cdot \cdot \cdot \cdot \; \phi_n(x)$$

of degrees 0, 1, 2, ... n, respectively, each of the form

$$\phi_r(x) = b_0 + b_1 x + b_2 x^2 + \cdot \quad \cdot \cdot \cdot \, b_r x^r, \tag{21}$$

and such that

$$
\left\{
\begin{aligned}
&(a) \sum_{-p}^{p} C_x \phi_m(x)\phi_n(x) = 0 && \text{for } m \neq n \\
&(b) \sum_{-p}^{p} C_x \left[\phi_r(x)\right]^2 = S_r \neq 0,
\end{aligned}
\right.
\tag{22}
$$

the value of S_r to be determined later.

We wish to approximate by means of these functions a function of x,

$$(23)\quad y = y'C_x = \left[a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x)\right]C_x,$$

which will be the equation of a curve fitting the given data.

Multiplying each side of (23) by $\phi_r(x)$ and summing from $x = -p$ to $+p$, — applying (22),—we have

$$(24)\quad \sum_{-p}^{p} y_x \phi_r(x) = a_r \sum_{-p}^{p} C_x\left[\phi_r(x)\right]^2 = a_r S_r.$$

Substitute in the left member of (24) the value of $\phi_r(x)$ in (21) and we have

$$b_0\sum_{-p}^{p} y_x + b_1\sum_{-p}^{p} y_x x + b_2\sum_{-p}^{p} y_x x^2 + \quad . \; . \; . + b_r\sum_{-p}^{p} y_x x^r = a_r S_r,$$

which, by (3), becomes

$$(25)\quad a_r = \frac{b_0M_0 + b_1M_1 + b_2M_2 + \cdots\cdots + b_rM_r}{S_r}.$$

This value of a_r is identical with $a_{r,r}$ found by the first method as may be shown in the following manner:

Let

$$(26)\quad J = \sum_{-p}^{p} C_x\left[y'_x - a_0\phi_0(x) - a_1\phi_1(x) - \cdots - a_n\phi_n(x)\right]^2,$$

which is to be minimized, where $y'_x C_x = y_x$.

Taking the partial derivative of J with respect to a_r, we have

$$\frac{\partial J}{\partial a_r} = -2\sum_{-p}^{p} C_x \phi_r(x)\left[y'_x - a_0\phi_0(x) - a_1\phi_1(x) \cdot \quad - a_n\phi_n(x)\right] = 0,$$

which reduces, by (22), to

$$(27) \qquad \sum_{-p}^{p} y'_x C_x \phi_r(x) - a_r \sum_{-p}^{p} C_x \phi_r^2(x) = 0,$$

or $\quad a_r S_r = \sum_{-p}^{p} y_x \phi_r(x) \qquad$ as in (24)

We are, therefore, able to write the values of $\frac{b_i}{S_r}$ in (25) by comparing the coefficients of the moments, M_i, in (25) with those in $a_{r,r}$ in equations (9) to (14), or in (16).

Thus, for $r = 4$, we have

$$a_{4,4} = C_4 M_0 + E_4 M_2 + F_4 M_4,$$

$$a_4 = \frac{b_0 M_0}{S_4} + \frac{b_1 M_1}{S_4} + \frac{b_2 M_2}{S_4} + \frac{b_3 M_3}{S_4} + \frac{b_4 M_4}{S_4},$$

$$(28) \qquad \therefore \frac{b_0}{S_4} = C_4, \ \frac{b_2}{S_4} = E_4, \ \frac{b_4}{S_4} = F_4,$$

$$\frac{b_1}{S_4} = \frac{b_3}{S_4} = 0.$$

$$b_0 = C_4 S_4, \qquad b_2 = E_4 S_4, \qquad b_4 = S_4 F_4,$$

$$(29) \qquad \therefore \phi_4(x) = S_4 C_4 + S_4 E_4 x^2 + S_4 F_4 x^4.$$

Now the value of $S_r = \sum_{-p}^{p} C_x \left[\phi_r(x)\right]^2$ can be found by comparing the coefficients of $C_x x^r M_r$ in the expansions given by the two different methods. In case IV, we have

$$y = (\cdot \quad \cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot + a_{4,4} x^4)\, C_x$$

$$= (F_4 M_4 x^4 + E_4 M_2 x^4 + C_4 M_0 x^4 + \text{terms of lower degree in } x)\, C_x \;,$$

so the desired coefficient is F_4.

By (23), we have

$$y = a_4 \phi_4(x) C_x + a_3 \phi_3(x) C_x + \cdots$$

$$= \left(\frac{b_0}{S_4} M_0 + \cdots\cdots + \frac{b_4}{S_4} M_4\right)(S_4 F_4 x^4 + S_4 E_4 x^2 + S_4 C_4) C_x + \cdots$$

$$= \frac{b_4}{S_4} M_4 \cdot S_4 F_4 x^4 C_x + \cdots\cdots$$

But by (28) $b_4 = S_4 F_4$ and the desired coefficient is

$$\frac{S_4 F_4}{S_4} \cdot S_4 F_4 = S_4 F_4^{\,2}.$$

Equating these coefficients, we have

$$S_4 F_4^{\,2} = F_4 \,,$$

(30) $\quad \therefore \; S_4 = \dfrac{1}{F_4}$.

Therefore, S_r is equal to the reciprocal of the coefficient of M_r in $a_{r,r}$. But examination of (16) shows that this coefficient is $\dfrac{P(r-2)}{P(r)}$.

$$(31) \qquad \therefore S_r = \frac{P(r)}{P(r-2)} .$$

The coefficient of x^i in $\phi_r(x)$ can now be found by multiplying the coefficient of M_i in $a_{r,r}$ by $\frac{P(r)}{P(r-2)} = S_r$. Indeed, it is possible to express $\phi_r(x)$ as the quotient of two determinants The coefficient of x^i in $P(r,x)$ will be identical with that of M_i in $P(r,M)$. We may, therefore, write

$$(32) \qquad \phi_r(x) = \frac{P(r,x)}{P(r)} \cdot \frac{P(r)}{P(r-2)} = \frac{P(r,x)}{P(r-2)} .$$

Proceeding in this manner, we obtain

$$\phi_0(x) = 1,$$

$$\phi_1(x) = x,$$

$$\phi_2(x) = x^2 - \frac{p}{2},$$

$$\phi_3(x) = x^3 - \frac{3p-1}{2}x,$$

$$\phi_4(x) = x^4 - (3p-2)x^2 + \frac{3p(p-1)}{4},$$

$$(33) \quad \phi_5(x) = x^5 - 5(p-1)x^3 + \frac{(15p^2-25p+6)x}{4},$$

$$\phi_6(x) = x^6 - 5\left(\frac{3p-4}{2}\right)x^4 + \frac{45p^2-105p+46}{4}x^2 - \frac{15p(p-1)(p-2)}{8},$$

$$\phi_7(x) = x^7 - 7\frac{3p-5}{2}x^5 + \frac{105p^2-315p+196}{4}x^3$$

$$-\left(\frac{105p^3-420p^2+441p-90}{8}\right)x,$$

$$\phi_8(x) = x^8 - 14(p-2)x^6 + \frac{7(15p^2-55p+44)}{2}x^4$$

$$-\frac{105p^3-525p^2+742p-264}{2}x^2$$

$$+\frac{105}{16}p(p-1)(p-2)(p-3).$$

The coefficient, a_i, of $\phi_i(x)$ is given below. In addition to the values obtained from (9) to (14), a_7 and a_8 have been added. The values of $A_1, B_2, \cdots\cdots J_6$ are given in the tables at the end of this paper.

$$a_0 = M_0,$$

$$a_1 = A_1M_1,$$

$$a_2 = B_2M_0 + C_2M_2,$$

$$a_3 = B_3M_1 + C_3M_3,$$

(34) $$a_4 = C_4M_0 + E_4M_2 + F_4M_4,$$

$$a_5 = C_5M_1 + E_5M_3 + F_5M_5,$$

$$a_6 = D_6M_0 + G_6M_2 + I_6M_4 + J_6M_6,$$

$$a_7 = D_7M_1 + G_7M_3 + I_7M_5 + J_7M_7,$$

$$a_8 = A_8M_0 + B_8M_2 + C_8M_4 + D_8M_6 + E_8M_8$$

where

$$D_7 = -\frac{8(105p^3 - 420p^2 + 441p - 90)}{d_7},$$

$$G_7 = \frac{16(105p^2 - 315p + 196)}{d_7},$$

$$I_7 = -\frac{224(3p-5)}{d_7},$$

$$J_7 = \frac{64}{d_7},$$

$$d_7 = 315p(p-1)(p-2)(p-3)(2p-1)(2p-3)(2p-5),$$

and

$$A_8 = \frac{2}{3(2p-1)(2p-3)(2p-5)(2p-7)},$$

$$B_8 = \frac{-16(105p^3 - 525p^2 + 742p - 264)}{d_8},$$

$$C_8 = \frac{112(15p^2 - 55p + 44)}{d_8},$$

$$D_8 = \frac{-448(p-2)}{d_8},$$

$$E_8 = \frac{32}{d_8},$$

$$d_8 = 315p(p-1)(p-2)(p-3)(2p-1)(2p-3)(2p-5)(2p-7).$$

It was shown above how the coefficients of x^i in $\phi_r(x)$ could be found from those of M_i in a_{rr}. It is evident from that, that with $\phi_r(x)$ known, the value of a_r can be immediately determined by changing x^i to M_i and multiplying the result by $F_r = \frac{P_{(r-2)}}{P_r}$. It will be shown later that these ϕ's can be determined independent of the determinants previously used, and more easily. a_7 and a_8 were determined from $\phi_7(x)$ and $\phi_8(x)$, respectively, and then checked by means of (16).

IV. PROPERTIES OF THE ϕ FUNCTION.

The similarity between the ϕ functions just derived and the Hermite polynomials, of which these may be said to be the analog, is evident, and leads one to expect that there must be a generating function analogous $e^{-x^2/2}$ by means of which all of the ϕ's can be found.' Also, one would naturally expect to find a recurrence formula and a second order difference equation analogous to the relationships existing between the Hermite polynomials.

This proves to be true. We have, in fact,

$$(35) \quad C_x \phi_n(x) = \left(-\frac{1}{2}\right)^n \Delta^n C_x (p+x)^{(n)}, \qquad \text{where}$$

$$(p+x)^{(n)} = (p+x)(p+x-1)(p+x-2)\cdots(p+x-n+1).$$

Expand the right member of (35) and divide both sides by C_x, and we obtain

$$(36) \quad \begin{aligned} \phi_n(x) = \left(-\frac{1}{2}\right)^n \Big[& (p-x)^{(n)} - n(p-x)^{(n-1)}(p+x) \\ & + \binom{n}{2}(p-x)^{(n-2)}(p+x)^{(2)} \cdots + (-1)^{n-1}(p-x)(p+x)^{(n-1)} + (-1)^n (p+x)^{(n)} \Big]. \end{aligned}$$

A study of the expression in the bracket brings out the following facts:

(37) *The coefficient of* x^n *is* $(-2)^n$, for it is seen to be equal to

$$(-1)^n - \binom{n}{1}(-1)^{n-1}(1) + \binom{n}{2}(-1)^{n-2}(1)^2 - \binom{n}{3}(-1)^{n-3}(1)^3 + \cdots$$

$$(-1)^n(1)^n = (-1)^n\left[1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots \binom{n}{n}\right] = (-1)^n 2^n.$$

(38). *If* n *is even, only even powers of* x *appear in the expanded form of the bracket; for* n *odd, only odd powers of* x *will occur.*

Consider the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}$, etc. With n even, the terms of odd degree in x in $(p-x)^{(n)}$ will be negative, and cancel the corresponding terms in the last parenthesis, $(p+x)^{(n)}$. If n is odd, the factor $(-1)^n$ causes the corresponding terms of even degree in the same two expansions to have opposite signs, and, therefore, to vanish. Similar reasoning holds for every pair of products in the bracket which have the equal coefficients, $\binom{n}{r}=\binom{n}{n-r}$. If n is odd, this will include every term. If n is even, the middle term will be

$$(p-x)^{(n/2)}(p+x)^{(n/2)}=(p-x)(p+x)(p-x-1)(p+x-1)(p-x-2)(p+x-2)\ldots,$$

the odd powers of x having zero coefficients.

Therefore, either all terms are of even, or all are of odd degree.

It is necessary first to prove that the relations (22) hold; that is,

$$\text{(a)}\ \sum_{-p}^{p} C_x\phi_m(x)\phi_n(x)=0,\qquad \text{for } m\neq n.$$

$$\text{(b)}\ \sum_{-p}^{p} C_x\left[\phi_m(x)\right]^2=S_m=\frac{m!(2p)^{(m)}}{2^{2m}}.$$

To prove the first relationship, we may proceed as follows:

Let

$$\text{(39)}\quad \begin{aligned}\sum_{-p}^{p} C_x\phi_m(x)\phi_n(x)&=\sum_{-p}^{p+1}\phi_m(x)\,C_x\phi_n(x)\\ &=\Delta^{-1}\Big[\phi_m(x)\cdot C_x\phi_n(x)\Big]_{p}^{p+1},\end{aligned}$$

where we may assume without loss of generality that $n>m$, for if it is not, $\phi_m(x)$ and $\phi_n(x)$ can be interchanged. Using the formula for finite integration,

$$\Delta^{-1} u_x v_x = u_x \Delta^{-1} v_x - \Delta u_x \Delta^{-2} v_{x+1} + \Delta^{2} u_x \Delta^{-3} v_{x+2} - \Delta^{3} u_x \Delta^{-4} v_{x+3} + \cdots\cdots,$$

we have

$$\Delta^{-1}\phi_m(x)\cdot C_x\phi_n(x) = \phi_m(x)\Delta^{-1}C_x\phi_n(x) - \Delta\phi_m(x)\Delta^{-2}C_{x+1}\phi_n(x+1)$$

$$+\Delta^{2}\phi_m(x)\Delta^{-3}C_{x+2}\phi_n(x+2)\,..\,+(-1)^m\Delta^m\phi_m(x)\Delta^{-m-1}C_{x+m}\phi_n(x+m)$$

(40)

$$+(-1)^{m+1}\Delta^{m+1}\phi_m(x)\Delta^{-m-2}C_{x+m+1}\phi_n(x+m+1)+\cdots\cdots$$

between the limits $-p$ and $p+1$.

Now $\Delta^{-1}C_x\phi_n(x) = \Delta^{-1}\left[\left(-\frac{1}{2}\right)^n\Delta^n C_x(p+x)^{(n)}\right]$, (by 35)

$$=\left(-\frac{1}{2}\right)^n\Delta^{n-1}C_x(p+x)^{(n)}$$

$$=\left(-\frac{1}{2}\right)^n C_x\Big[(p-x)^{(n-1)}(p+x)-\binom{n-1}{1}(p-x)^{(n-2)}(p+x)^{(2)}$$

$$+\binom{n-1}{2}(p-x)^{(n-3)}(p+x)^{(3)}\cdots\cdot(-1)^{n-1}(p+x)^{(n)}\Big]$$

$$=\left(-\frac{1}{2}\right)^n C_x(p+x)\Big[f\{(p-x),(p+x-1)\}\Big]$$

$$=\left(-\frac{1}{2}\right)^n\frac{(2p)!}{2^{2p}(p+x)!(p-x)!}\cdot(p+x)\Big[f\{(p-x)(p+x-1)\}\Big]$$

For the lower limit, $-p$, the factor $p+x=0$, and for $x=p+1, \frac{1}{(p-x)!}=\frac{1}{(-1)!}=\frac{1}{\infty}=0.$

$$\therefore \Delta^{-1}C_x\phi_n(x)\Big|_{-p}^{p+1}=0$$

Similarly, each of the succeeding terms up to and including the term $\Delta^m\phi_m(x)\Delta^{-m-1}C_{x+m}\phi_n(x+m)$ becomes identically zero because of the factor $p+x$ or C_x for the lower and upper limits, respectively.

Since $\phi_m(x)$ is of the m-th degree in x, $\Delta^{m+1}\phi_m(x)$ and all higher differences are identically zero.

$$(41) \quad \therefore \sum_{-p}^{p} C_x\phi_m(x)\phi_n(x)=0, \qquad \text{for} \quad m\neq n.$$

If $m=n$, the first m terms vanish as in the preceding case. The $(m+1)th$ term is the only term left in the series.

$$(42) \quad \therefore \sum_{-p}^{p} C_x\phi_m(x)\phi_m(x)=(-1)^m\Delta^m\phi_m(x).\Delta^{-m-1}C_{x+m}\phi_m(x+m)$$

$$=(-1)^m m!\,\Delta^{-m-1}C_{x+m}\phi_m(x+m).$$

But

$$\Delta^{-m-1}C_{x+m}\phi_m(x+m)=\Delta^{-m-1}\left\{\left(-\tfrac{1}{2}\right)^m\Delta^m C_{x+m}(p+x+m)^{(m)}\right\}$$

$$=\left(-\tfrac{1}{2}\right)^m\Delta^{-1}\,\frac{(2p)!\,(p+x+m)^{(m)}}{2^{2p}\,(p+x+m)!\,(p-x-m)!}$$

(43)

$$= \frac{(-1)^m (2p)!}{2^{2p+m}} \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}.$$

It is now necessary to find a function, u_x, such that

$$\Delta u_x = \frac{1}{(p+x)!(p-x-m)!}.$$

Since $\Delta = (E-1)$, we may write

$$\Delta u_x = (E-1) \rightarrow u_x = \frac{1}{(p+x)!(p-x-m)!},$$

$$u_x = \frac{-1}{1-E} \rightarrow \frac{1}{(p+x)!(p-x-m)!}$$

$$= -\left[1+E+E^2+E^3+\cdots\right] \rightarrow \frac{1}{(p+x)!(p-x-m)!}.$$

(44) $\therefore \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}\Bigg|_{-p}^{p+1} = u_x\Bigg|_{-p}^{p+1} =$

$$-\frac{1}{(p+x)!(p-x-m)!} - \frac{1}{(p+x+1)!(p-x-m-1)!} - \frac{1}{(p+x+2)!(p-x-m-2)} + \cdots,$$

between the limits $-p$ and $p+1$. Substitution of the upper limit, $p+1$, makes every term zero, because $(p-x-m-k)! = \infty$.

For $x=-p$, the right member becomes

$$\frac{1}{0!(2p-m)!} + \frac{1}{1!(2p-m-1)!} + \frac{1}{2!(2p-m-2)!} + \cdots$$

$$+ \frac{1}{(2p-m-1)!\,[2p-m-(2p-m-1)]!} + \frac{1}{(2p-m)!\,0!},$$

all succeeding terms being zero because of the second factor.

$$\therefore \Delta^{-1} \frac{1}{(p+x)!(p-x-m)!}$$

$$= \frac{1}{(2p-m)!}\left[1+\binom{2p-m}{1}+\binom{2p-m}{2}+\cdots\binom{2p-m}{2p-m-1}+\binom{2p-m}{2p-m}\right]$$

or

$$(45) \quad = \frac{2^{2p-m}}{(2p-m)!}$$

Returning to (43) and then to (42), we have

$$(46) \quad \sum_{-p}^{p} C_x\left[\phi_m(x)\right]^2 = (-1)^m m! \cdot \frac{(-1)^m (2p)!}{2^{2p+m}} \cdot \frac{2^{2p-m}}{(2p-m)!}$$

$$= \frac{m!(2p)^{(m)}}{2^{2m}} = S_m, \text{ by } (22),$$

[By (31), $S_m = \dfrac{P_{(m)}}{P_{(m-2)}}$.

$$(47) \quad \therefore P_{(m)} = \frac{m!(2p)^{(m)}}{2^{2m}} P_{(m-2)}. \qquad \text{See } (17).]$$

Therefore, $C_x\phi_n(x) = \left(-\frac{1}{2}\right)^n \Delta^n C_x (p+x)^{(n)}$ satisfies both conditions of (22).

Recurrence Formula.

It is necessary to note that

$$\Delta C_x (p+x)^{(n)} = \frac{(2p)!}{2^{2p}} \left[\frac{(p+x+1)^{(n)}}{(p+x+1)!(p-x-1)!} - \frac{(p+x)^{(n)}}{(p+x)!(p-x)!} \right]$$

$$= C_x (p+x)^{(n-1)} \left[\frac{(p+x+1)(p-x)}{p+x+1} - (p+x-n+1) \right]$$

$$= C_x (p+x)^{(n-1)} (-2x+n-1). \tag{48}$$

$$\therefore \Delta^{n+1} C_x (p+x)^{(n+1)} = \Delta^n \left[\Delta C_x (p+x)^{(n+1)} \right]$$

$$= \Delta^n \left[C_x (p+x)^{(n)} (-2x+n) \right]$$

$$= (-2x+n) \Delta^n C_x (p+x)^{(n)} + n(-2) \Delta^{n-1} C_{x+1} (p+x+1)^{(n)}.$$

But $C_{x+1} (p+x+1)^{(n)} = (1+\Delta) C_x (p+x)^{(n)}.$

$$\therefore \Delta^{n+1} C_x (p+x)^{(n+1)} = (-2x+n) \Delta^n C_x (p+x)^{(n)}$$

$$-2n \Delta^n C_x (p+x)^{(n)} - 2n \Delta^{n-1} C_x (p+x)^{(n)}$$

$$= -2x \Delta^n C_x (p+x)^{(n)} - n \Delta^{n-1} \left[\Delta C_x (p+x)^{(n)} \right] - 2n \Delta^{n-1} C_x (p+x)^{(n)}$$

$$= -2x \Delta^n C_x (p+x)^{(n)} - n \Delta^{n-1} C_x (p+x)^{(n-1)} (-2x+n-1)$$

$$- n \Delta^{n-1} C_x (p+x)^{(n-1)} (2p+2x-2n+2)$$

$$= -2x \Delta^n C_x (p+x)^{(n)} - n(2p+1-n) \Delta^{n-1} C_x (p+x)^{(n-1)}.$$

Now $\Delta^n C_x (p+x)^{(n)} = (-2)^n C_x \phi_n (x),$

$$\therefore (-2)^{n+1} C_x \phi_{n+1}(x) = -2x(-2)^n C_x \phi_n(x) - n(2p+1\ n)(-2)^{n-1} C_x \phi_{n-1}(x),$$

or

$$(49) \qquad 4\phi_{n+1}(x) - 4x\phi_n(x) + n(2p+1-n)\phi_{n-1}(x) = 0.$$

Difference Equation.

To simplify the reductions later in this development, it is desirable to have the following identities:—

(50)

$$\text{(a)} \quad \Delta C_x = C_x\left[\frac{p-x}{p+x+1} - 1\right] = C_x \frac{-2x-1}{p+x+1},$$

$$\text{(b)} \quad \Delta^2 C_x = C_x \frac{4x^2+8x+2-2p}{(p+x+2)(p+x+1)},$$

$$\text{(c)} \quad C_{x+1} = C_x \frac{p-x}{p+x+1},$$

$$\text{(d)} \quad \Delta C_{x+1} = C_{x+1}\frac{-2x-3}{p+x+2} = C_x \frac{p-x}{p+x+1} \cdot \frac{-2x-3}{p+x+2},$$

$$\text{(e)} \quad C_{x+2} = C_x \frac{(p-x)(p-x-1)}{(p+x+2)(p+x+1)}.$$

Let $u_x = C_x(p+x)^{(n)}$,

$$\Delta^n u_x = \Delta^n C_x (p+x)^{(n)} = (-2)^n C_x \phi_n(x),$$

$$\Delta u_x = C_x (p+x)^{(n-1)}(-2x+n-1), \qquad \text{(by 48)}$$

and, multiplying both sides by $(p+x-n+1)$, we get

$$(p+x-n+1)\Delta u_x = (-2x+n-1)u_x.$$

Difference this equation $n+1$ times, having

$$(51) \qquad (p+x-n+1)\Delta^{n+2} u_x + (n+1)\Delta^{n+1}(u_{x+1}) = (-2x+n-1)\Delta^{n+1} u_x - 2(n+1)\Delta^n u_{x+1}$$

or, since $u_{x+1} = (1+\Delta) \rightarrow u_x$, we have

$$(52)\ (p+x-n+1)\Delta^{n+2}u_x+(n+1)\Delta^{n+2}u_x+(n+1)\Delta^{n+1}u_x$$

$$=(-2x+n-1)\Delta^{n+1}u_x-2(n+1)\Delta^{n+1}u_x-2(n+1)\Delta^{n}u_x.$$

Combining like differences, we get

$$(p+x+2)\Delta^{n+2}u_x+(2x+2n+4)\Delta^{n+1}u_x+2(n+1)\Delta^{n}u_x=0.$$

$$(53)\ (p+x+2)\Delta^2(-2)^nC_x\phi_n(x)+(2x+2n+4)\Delta(-2)^nC_x\phi_n(x)$$

$$+2(n+1)(-2)^nC_x\phi_n(x)=0.$$

This may be simplified by making the following reductions:

$$\Delta^2C_x\phi_n(x)=\phi_n(x)\cdot\Delta^2C_x+2\Delta\phi_n(x)\cdot\Delta C_{x+1}+\Delta^2\phi_n(x)\cdot C_{x+2}$$

$$=\phi_n(x)\cdot C_x\frac{4x^2+8x+2-2p}{(p+x+2)(p+x+1)}+2\Delta\phi_n(x)\cdot C_x\frac{(p-x)(-2x-3)}{(p+x+2)(p+x+1)}$$

$$+\Delta^2\phi_n(x)\cdot C_x\frac{(p-x)(p-x-1)}{(p+x+2)(p+x+1)},$$

$$\Delta C_x\phi_n(x)=\phi_n(x)\cdot\Delta C_x+\Delta\phi_n(x)\cdot C_{x+1}$$

$$=\phi_n(x)\cdot C_x\frac{-2x-1}{p+x+1}+\Delta\phi_n(x)\cdot C_x\frac{p-x}{p+x+1}.$$

Substituting these values in (53), noticing that C_x is a common factor, and that $\frac{1}{p+x+1}$ can be made a factor by multiplying the last term by $\frac{p+x+1}{p+x+1}$, we have

(54)

$$(p-x)(p-x-1)\Delta^2\phi_n(x)+\left[2(p-x)(-2x-3)+(p-x)(2x+2n+4)\right]\Delta\phi_n(x)$$

$$+\left[4x^2+8x+2-2p+(2x+2n+4)(-2x-1)+2(n+1)(p+x+1)\right]\phi_n(x)=0.$$

The coefficient of $\phi_n(x)$ reduces to $2n(p-x)$, and that of $\Delta\phi_n(x)$, to $(p-x)(2_n-2-2x)$. Dividing by $(p-x)$, we obtain the desired second order difference equation

$$(p-x-1)\Delta^2\phi_n(x)+2(n-1-x)\Delta\phi_n(x)+2n\phi_n(x)=0. \tag{55}$$

V. OTHER METHODS OF DERIVING THESE FUNCTIONS,—

3RD. METHOD.

Equations (2) were divided into two groups, (4) and (4a), from which these functions were developed. It would be expected that the functions could be derived from (2). This is easily seen to be true.

Let $Q_{(n)}$ be the determinant formed by the coefficients of $a_{n,i}$ in (2).

(56) Let
$$Q_{(n)}=\begin{vmatrix} m_0 & 0 & m_2 & 0 & m_4 & \cdots \\ 0 & m_2 & 0 & m_4 & 0 & \cdots \\ m_2 & 0 & m_4 & 0 & m_6 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & m_{2n} \end{vmatrix}$$

for $n > 0$ In the special case $n = -1$, we will define $Q_{(-1)} = 1$.

Let $$Q_{(n,x)} = \begin{vmatrix} m_0 & 0 & m_2 & 0 & \cdots & 1 \\ 0 & m_2 & 0 & m_4 & \cdots & x \\ m_2 & 0 & m_4 & 0 & \cdots & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & x^n \end{vmatrix}$$

Then $\phi_n(x)$ is easily seen to be

$$\phi_n(x) = \frac{Q_{(n,x)}}{Q_{(n-1,)}}. \tag{57}$$

Explicitly, we will have

$$\phi_0(x) = \frac{Q_{(0,x)}}{Q_{(-1)}} = 1,$$

$$\phi_1(x) = \frac{Q_{(1,x)}}{Q_{(0)}} = \frac{\begin{vmatrix} m_0 & 1 \\ 0 & x \end{vmatrix}}{m_0} = x,$$

$$\phi_2(x) = \frac{\begin{vmatrix} m_0 & 0 & 1 \\ 0 & m_2 & x \\ m_2 & 0 & x^2 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 \\ 0 & m_2 \end{vmatrix}} = x^2 \frac{-m_2^2}{m_0 m_2} = x^2 - \frac{p}{2},$$

$$\phi_3(x) = \frac{\begin{vmatrix} m_0 & 0 & m_2 & 1 \\ 0 & m_2 & 0 & x \\ m_2 & 0 & m_4 & x^2 \\ 0 & m_4 & 0 & x^3 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 \\ 0 & m_2 & 0 \\ m_2 & 0 & m_4 \end{vmatrix}} = x^3 + x\,\frac{(m_2^2 m_4 - m_4^2)}{m_2(m_0 m_4 - m_2^2)}$$

$$= x^3 - \frac{m_4}{m_2}x = x^3 - \frac{3p-1}{2}x,$$

and so on.

The coefficients of the ϕ's can be found from the formula

$$(58) \qquad a_i = \frac{Q_{(i,M)}}{Q_{(i)}},$$

where $Q_{(i,M)}$ is $Q_{(i)}$, the last column being replaced with M_0, $M_1, M_2, \cdots\cdots M_i$. Thus, we have

$$a_0 = \frac{M_0}{m_0} = M_0 ,$$

$$a_1 = \frac{\begin{vmatrix} m_0 & M_0 \\ 0 & M_1 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 \\ 0 & m_2 \end{vmatrix}} = \frac{m_0 M_1}{m_0 m_2} = \frac{2}{p} M_1 = A_1 M_1 ,$$

$$a_2 = \frac{\begin{vmatrix} m_0 & 0 & M_0 \\ 0 & m_2 & M_1 \\ m_2 & 0 & M_2 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 \\ 0 & m_2 & 0 \\ m_2 & 0 & m_4 \end{vmatrix}} = \frac{-m_2 M_0}{m_0 m_4 - m_2^2} + \frac{m_0 M_2}{m_0 m_4 - m_2^2}$$

$$= -\frac{2}{2p-1} M_0 + \frac{4}{p(2p-1)} M_2$$

$$= B_2 M_0 + C_2 M_2 ,$$

$$u_3 = \frac{\begin{vmatrix} m_0 & 0 & m_2 & M_0 \\ 0 & m_2 & 0 & M_1 \\ m_2 & 0 & m_4 & M_2 \\ 0 & m_4 & 0 & M_3 \end{vmatrix}}{\begin{vmatrix} m_0 & 0 & m_2 & 0 \\ 0 & m_2 & 0 & m_4 \\ m_2 & 0 & m_4 & 0 \\ 0 & m_4 & 0 & m_6 \end{vmatrix}} = \frac{M_1(m_2^2 m_4 - m_0 m_4^2) + M_3 m_2 (m_0 m_4 - m_2)^2}{m_6 m_2 (m_0 m_4 - m_2^2) + m_4^2 (m_0 m_4 - m_2^2)}$$

$$= \frac{-\frac{p(3p-1)}{4} M_1 + \frac{p}{2} M_3}{\frac{3}{16} p^2 (2p-1)(p-1)}$$

$$= B_3 M_1 + C_3 M_3.$$

This method becomes more difficult than the first because of the higher order determinants, and is, therefore, of less value in deriving the functions.

A fourth method of developing these polynomials is to build up a set of orthogonal functions in the following manner:[1]

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[1] This method is given in a thesis by Harry R. Mathias, entitled "Properties of Orthogonal and Biorthogonal Functions from the Standpoint of Integral Equations," written at Indiana University August, 1925. He cites as his reference E. Goursat—"Recherches sur les equations intégrales linéaires," Ann. de la Fac. De Toulouse, t. 10, 2nd series, 1908, pp. 5-98, especially page 66.

Assume a set of functions

$f_0(x)=1,\ f_1(x)=x,\ f_2(x)=x^2,\ f_3(x)=x^3,\ f_4(x)=x^4.$

It is required to find a set of functions, $\phi_i(x)$, such that

$$\sum_{-p}^{p} C_x\phi_m(x)\phi_n(x)=0, \qquad \text{for } m\neq n.$$

Let $\phi_0(x)=f_0(x)=1.$

We may then form the equations

$$\sum_{-p}^{p} C_x\left[f_i(x)-a_i f_0(x)\right]f_0(x)=0$$

, obtaining

$$\sum_{-p}^{p} C_x\left[x-a_1\cdot 1\right]1=0,$$

$$\sum_{-p}^{p} C_x\left[x^2-a_2\cdot 1\right]1=0,$$

(59)
$$\sum_{-p}^{p} C_x\left[x^3-a_3\cdot 1\right]1=0,$$

$$\sum_{-p}^{p} C_x\left[x^4-a_4\cdot 1\right]1=0,$$

the solutions of which are

$$a_1=0,\ a_2=m_2=\frac{p}{2},\ a_3=0,\ a_4=m_4=\frac{p(3p-1)}{4}.$$

Let $\theta_1(x) = x - a_1 \cdot 1 = x = \phi_1(x),$

$$\theta_2(x) = x^2 - \frac{p}{2},$$

$$\theta_3(x) = x^3,$$

$$\theta_4(x) = x^4 - \frac{p(3p-1)}{4}.$$

Form a set of equations, similar to (59), using the Θ's .

$$\sum_{-p}^{p} C_x\left(x^2 - \frac{p}{2} - b_1 x\right)x = 0,$$

$$\sum_{-p}^{p} C_x(x^3 - b_2 x)x = 0,$$

$$\sum_{-p}^{p} C_x\left(x^4 - \frac{p(3p-1)}{4} - b_3 x\right)x = 0.$$

From these equations, we get

$$b_1 = 0, \qquad b_2 = \frac{3p-1}{2}, \qquad b_3 = 0.$$

Let $\psi_2(x) = \left(x^2 - \frac{p}{2}\right) = \phi_2(x),$

$$\psi_3(x) = \left(x^3 - \frac{3p-1}{2}x\right),$$

$$\psi_4(x) = \left(x^4 - \frac{p(3p-1)}{4}\right),$$

Similarly, the equations

$$\Sigma C_x\left[x^3 - \frac{3p-1}{2}x - C_1\left(x^2 - \frac{p}{2}\right)\right]\left(x^2 - \frac{p}{2}\right) = 0,$$

$$\Sigma C_x\left[x^4 - \frac{p(3p-1)}{4} - C_2\left(x^2 - \frac{p}{2}\right)\right]\left(x^2 - \frac{p}{2}\right) = 0,$$

lead to the values

$$C_1 = 0, \qquad C_2 = (3p-2).$$

C_1 is easily seen to equal 0, as the term independent of C_1 is $\Sigma \{ C_x$ multiplied by an odd function in x. Expanding the second equation, we get

$$C_2\left[\sum_{-p}^{p} C_x\left(x^2-\frac{p}{2}\right)^2\right] = \sum_{-p}^{p} C_x\left[x^4-\frac{p(3p-1)}{4}\right]\left(x^2-\frac{p}{2}\right),$$

$$C_2 = \frac{\sum_{-p}^{p} C_x\left[x^6-\frac{p}{2}x^4-\frac{p(3p-1)}{4}x^2+\frac{p^2}{8}(3p-1)\right]}{\sum_{-p}^{p} C_x\left[x^4-px^2+\frac{p^2}{4}\right]}$$

$$= \frac{m_6-\frac{p}{2}m_4-\frac{p(3p-1)}{4}m_2+\frac{p^2}{8}(3p-1)m_0}{m_4-pm_2+\frac{p^2}{4}m_0},$$

and substituting the values of the moments, we reduce C_2 to the value above, $C_2 = (3p-2)$.

Let $\lambda_3(x) = x^3 - \frac{3p-1}{2}x = \phi_3(x)$,

$$\lambda_4(x) = x^4 - \frac{p(3p-1)}{4} - (3p-2)\left(x^2-\frac{p}{2}\right)$$

$$= x^4 - (3p-2)x^2 + \frac{3p(p-1)}{4},$$

This method may be continued to obtain as many of the ϕ's as desired. It is to be noted that $\lambda_4(x)$ is $\phi_4(x)$. However, the proof of this would necessitate adding another function, $f_5(x) = x^5$, at the beginning, and carrying out the process with another equation in each set.

Method 5.

Gram's equation, number 18, is

$$\phi'_m(x) = n^{(m)} - \binom{m}{1}(n-1)^{(m-1)}\cdot 2x + \binom{m}{2}(n-2)^{(m-2)}\cdot 2^2 x^{(2)} + \cdots \quad (1)$$

This gives ϕ's for the origin at the left end of the distribution, the variates running from 0 to n. Let this equation be transformed as follows:

Let $m = n$,

$x = x' + p$, the primes being dropped,

$n = 2p$

$$\phi'_n(x) = (2p)^{(n)} - \binom{n}{1}(2p-1)^{(n-1)} 2(p+x) + \binom{n}{2}(2p-2)^{(n-2)}\cdot 2^2 (p+x)^{(2)}$$

If the values of $n = 0, 1, 2, \ldots$ . . be substituted, we will have $\phi_n(x) = \left(-\frac{1}{2}\right)^n \phi'_n(x)$ and the required functions are found again.

VI. Example.

As an example to illustrate the use of this development, I have chosen one used by Karl Pearson,[2] the data of which he attributes to T. N. Thiele,[3] and are the frequencies in a game of "patience." On page 295, Vol. I, Pearson states, "Now either of the curves in Illustration I and II is a good example of the impossibility of using the method of least squares for systematic curve fitting." (The set of data below is his illustration II.)

In volume 2, using the Method of Moments, he fits curves of

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[1] Gram uses the notation $n^{m|-1}$ instead of $n^{(m)}$, and $(m)_r$ for $\binom{m}{r}$.

[2] On the Systematic Fitting of Curves to Observations and measurements. Biometrika, vol. 1, (1902), pp. 265-303; vol. (1903) pp/1-27.

[3] Thiele—Forelaesninger over Almindelig Iagttagelseslaere, Copenhagen, (1889) p. 12.

the second to the sixth degree to this data. Noting that his statement, page 18, "taking the sixth parabola as the best fit" is correct, it is found that the sum of the squares of the errors is more than 1400. His results for the skew frequency curve gives as the sum of the squares of the errors, 782, which indicates the second curve to be a more accurate fit.

Let the given data be

Value of Character	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	0	0	3	7	35	101	89	94	70	46	30	15	4	5	1
Class Marks (x)	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7

The mean of this distribution being 11.86, the origin has been chosen as at 12; p, therefore, is 7.

Suppose we wish to find the curve

$$y=\Big[a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)$$

$$+a_4\phi_4(x)+a_5\phi_5(x)+a_6\phi_6(x)\Big]C_x$$

The values of the coefficients, a_i , are

$a_0=M_0,$

$a_1=A_1M_1,$

$a_2=B_2M_0+C_2M_2,$

$a_3=B_3M_1+C_3M_3,$

$a_4=C_4M_0+E_4M_2+F_4M_4,$

$a_5=C_5M_1+E_5M_3+F_5M_5,$

$a_6=D_6M_0+G_6M_2+I_6M_4+J_6M_6.$

The moments, $M_i = \sum_{-p}^{p} y_x x^i$, are computed and found to be

$$M_0 = 500, \qquad M_3 = 1466,$$

$$M_1 = -70, \qquad M_4 = 26{,}664,$$

$$M_2 = 2088, \qquad M_5 = 64{,}010,$$

$$M_6 = 607{,}368.$$

From Table I, for $p = 7$, we find

$A_1 = 0.28571\ 42857$, $\quad C_5 = .(2)50260\ 85026$,

$B_2 = -0.15384\ 61538$, $\quad E_5 = -.(2)10656\ 01066$,

$C_2 = 0.043956\ 04396$, $\quad F_5 = (4)35520\ 03552$,

$B_3 = -0.048840\ 04884$, $\quad D_6 = -.(2)10360\ 01036$,

$C_3 = (2)48840\ 04884$, $\quad G_6 = .(3)99719\ 21083$,

$C_4 = .(1)13986\ 01399$, $\quad I_6 = -.(3)11182\ 23340$.

$E_4 = -.(2)84360\ 08436$, $\quad J_6 = .(5)26311\ 13742$.

$F_4 = .(3)44400\ 04440$,

Substituting in the above equations, and putting $p = 7$ in $\phi_i(x)$ we have

$$\phi_0(x) = 1,$$

$$\phi_1(x) = x,$$

$$\phi_2(x) = x^2 - 3.5,$$

$$\phi_3(x) = x^3 - 10x,$$

$$\phi_4(x) = x^4 - 19x^2 + 31.5,$$

$$\phi_5(x) = x^5 - 30x^3 + 141.5x,$$

$$\phi_6(x) = x^6 - 42.5x^4 + 379x^2 - 393.75.$$

e corresponding values, and collecting like have

$$36 - 69.80444x + 60.15802x^2 - 1.29045x^3$$

$$6x^4 + 0.39564x^5 + 0.18056x^6) C_x.$$

attention here to the fact, that if it is desired of the second, third, fourth, or fifth degree, be used, without change, using only as many egree required. Thus for the fourth degree

$${}_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + a_4\phi_4(x)\,) C_x,$$

$$-125.78750x - 8.27422x^2 + 10.57875x^3$$

$$+ 1.21744x^4) C_x.]$$

Substituting values of x from -7 to $+7$ in the above equation, we obtain the following results:

x	y'	C_x (From Table II)	$y = y'C_x$	y to nearest integer	Observed y_x	Error $y - y_x$
−7	3385.714	.(4)61035 156	.20	0	0	0
−6	258.775	.(3)85449 219	.22	0	0	0
−5	− 20.817	.(2)55541 992	−.12	0	3	− 3
−4	421.199	.(1)22216 797	9.36	9	7	2
−3	713.454	.(1)61096 191	43.59	44	35	9
−2	701.412	0.12219 238	85.71	86	101	−15
−1	539.835	0.18328 857	98.94	99	89	10
0	415.254	0.20947 266	86.98	87	94	− 7
1	398.437	0.18328 857	73.02	73	70	3
2	426.868	0.12219 238	52.16	52	46	6
3	417.224	(1)61096 191	25.49	*26	30	− 4
4	507.857	.(1)22216 797	11.28	11	15	− 4
5	1431.276	.(2)55541 992	7.95	8	4	4
6	5016.640	.(3)85449 219	4.28	4	5	− 1
7	14822.246	.(4)61035 156	.90	1	1	0

The sum of the squares of the errors for this curve is 562, as compared with more than 1400 in Pearson's first method, and 782 for his skew frequency curve.

The fourth degree curve, found by this method, gives 1170 for the sum of the squares of the errors.

*Taken as 26 so that $\Sigma y = \Sigma y_x = 500$.

TABLE I

(Number in parenthesis indicates the number of ciphers between the decimal point and the first significant figure.)

p	A_1	A_2	B_2	C_2	p
1	2.00000 00000	2.00000 00000	- 2.00000 00000	4.00000 00000	1
2	1.00000 00000	1.66666 66667	- 0.66666 66667	0.66666 66667	2
3	0.66666 66667	1.60000 00000	- 0.40000 00000	0.26666 66667	3
4	0.50000 00000	1.57142 85714	- 0.28571 42857	0.14285 71429	4
5	0.40000 00000	1.55555 55556	- 0.22222 22222	.(1)88888 88889	5
6	0.33333 33333	1.54545 45455	- 0.18181 81818	.(1)60606 06061	6
7	0.28571 42857	1.53846 15385	- 0.15384 61538	.(1)43956 04396	7
8	0.25000 00000	1.53333 33333	- 0.13333 33333	.(1)33333 33333	8
9	0.22222 22222	1.52941 17647	- 0.11764 70588	.(1)26143 79085	9
10	0.20000 00000	1.52631 57895	- 0.10526 31579	.(1)21052 63157	10
11	0.18181 81818	1.52380 95230	-(1)95238 09524	.(1)17316 01732	11
12	0.16666 66667	1.52173 91304	-(1)86956 52174	.(1)14492 75362	12
13	0.15384 61538	1.52000 00000	-(1)80000 00000	.(1)12307 69231	13
14	0.14285 71429	1.51851 85185	-(1)74074 07407	.(1)10582 01058	14
15	0.13333 33333	1.51724 13793	-(1)68965 44828	(2)91954 02298	15
16	0.12500 00000	1.51612 90322	-(1)64516 12903	.(2)80645 16129	16
17	0.11764 70588	1.51515 15152	-(1)60606 06061	.(2)71301 24778	17
18	0.11111 11111	1.51428 57143	-(1)57142 85714	.(2)63492 06349	18
19	0.10526 31579	1.51351 35135	-(1)54054 05405	.(2)56899 00427	19
20	0.10000 00000	1.51282 05128	-(1)51282 05128	.(2)51282 05128	20

p	A_3	B_3	C_3	p
2	3.77777 77778	- 1.11111 11111	0.44444 44444	2
3	2.08888 88889	- 0.35555 55556	.(1)88888 88889	3
4	1.46031 74602	- 0.17460 31746	.(1)31746 03175	4
5	1.12592 59259	- 0.10370 37037	.(1)14814 81481	5
6	0.91717 17172	-.(1)68686 86869	(2)80808 08081	6
7	0.77411 47741	-.(1)48840 04884	.(2)48840 04884	7
8	0.66984 12698	-.(1)36507 93651	.(2)31746 03175	8
9	0.59041 39434	-.(1)28322 44009	.(2)21786 49238	9
10	0.52787 52437	-.(1)22612 08577	.(2)15594 54190	10
11	0.47734 48773	-.(1)18470 41847	.(2)11544 01154	11
12	0.43566 09574	-.(1)15371 10232	.(3)87834 87044	12
13	0.40068 37607	-.(1)12991 45299	.(3)68376 06838	13
14	0.37091 30375	-.(1)11124 67779	.(3)54266 72093	14
15	0.34526 54625	-.(2)96332 78599	.(3)43787 62999	15
16	0.32293 90681	- (2)84229 39068	.(3)35842 29391	16
17	0.30332 73916	-.(2)74272 13310	.(3)29708 85324	17
18	0.28596 32742	-.(2)65981 94834	(3)24898 84843	18
19	0.27048 10073	-.(2)59006 37480	(3)21073 70528	19
20	0.25659 01934	- (2)53081 42149	(3)17993 70220	20

ρ	A_4	B_4	C_4	ρ
2	2.66666 66667	− 3.33333 33333	0 66666 66667	2
3	2.20000 00000	− 1.33333 33333	0.13333 33333	3
4	2.08571 42857	− 0.85714 28571	.(1)57142 85714	4
5	2.03174 60317	− 0 63492 63492	.(1)31746 03175	5
6	2.00000 00000	− 0.50505 05051	.(1)20202 02020	6
7	1.97902 09790	− 0.41958 04196	.(1)13986 01399	7
8	1.96410 25641	− 0.35897 43590	.(1)10256 41025	8
9	1.95294 11765	− 0.31372 54902	.(2)78431 37255	9
10	1.94427 24458	− 0.27863 77709	.(2)61919 50464	10
11	1.93734 33584	− 0.25062 65664	.(2)50125 31328	11
12	1.93167 70186	− 0.22774 32712	.(2)41407 86749	12
13	1.92695 65217	− 0.20869 56522	.(2)34782 60870	13
14	1.92296 29630	− 0.19259 25926	.(2)29629 62963	14
15	1.91954 02299	− 0.17879 94891	.(2)25542 78416	15
16	1.91657 39711	− 0.16685 20578	.(2)22246 94105	16
17	1.91397 84946	− 0.15640 27370	.(2)19550 34213	17
18	1.91168 83117	− 0.14718 61472	.(2)17316 01732	18
19	1.90965 25096	− 0.13899 61390	.(2)15444 01544	19
20	1.90783 09078	− 0.13167 01317	.(2)13860 01386	20

ρ	D_4	E_4	F_4	ρ
2	7.77777 77778	− 1.77777 77778	0.44444 44444	2
3	1.71851 85185	− 0.20740 74074	.(1)29629 62963	3
4	0.77777 77778	− .(1)63492 06349	.(2)63492 06349	4
5	0.44656 08466	− .(1)27513 22751	.(2)21164 02116	5
6	0.29046 01571	− .(1)14365 88103	.(3)89786 75645	6
7	0.20424 02042	− .(2)84360 08436	.(3)44400 04440	7
8	0.15152 62515	− .(2)53724 05372	.(3)24420 02442	8
9	0.11692 08424	− .(2)36310 82062	.(3)14524 32824	9
10	.(1)92970 98957	− .(2)25685 12785	.(4)91732 59947	10
11	.(1)75704 41255	− .(2)18834 96620	.(4)60757 95549	11
12	.(1)62843 75850	− .(2)14220 88379	.(4)41826 12878	12
13	.(1)53006 31735	− .(2)10999 62840	.(4)29728 72538	13
14	.(1)45312 71198	− .(3)86826 75349	.(4)21706 68837	14
15	.(1)39181 82002	− .(3)69735 85518	.(4)16217 64074	15
16	.(1)34217 03127	− .(3)56853 29378	.(4)12359 41169	16
17	.(1)30140 11078	− .(3)46959 15512	.(5)95835 01045	17
18	.(1)26751 17185	− .(3)39234 54904	.(5)75451 05584	18
19	.(1)23903 60285	− .(3)33115 82259	.(5)60210 58653	19
20	.(1)21487 88465	− .(3)28206 34400	.(5)48631 62758	20

p	A_5	B_5	C_5	p
3	5.31555 55556	− 2.31111 11111	0.19555 55556	3
4	3.15206 34920	− 0.86984 12698	.(1)46349 20635	4
5	2.28176 36680	− 0.46490 29982	.(1)18059 96473	5
6	1.79717 17172	− 0.29090 90909	.(2)88888 88889	6
7	1.48530 58058	− 0.19962 25996	.(2)50260 85026	7
8	1.26686 60968	− 0.14562 47456	.(2)31176 23118	8
9	1.10499 84438	− 0.11098 66169	.(2)20666 04420	9
10	0.98009 86125	−.(1)87421 16730	.(2)14402 01812	10
11	0.88072 97705	−.(1)70654 75135	.(2)10436 86658	11
12	0.79975 28076	−.(1)58297 25830	.(3)78047 55631	12
13	0.73247 36327	−.(1)48925 37414	.(3)59889 86859	13
14	0.67567 56659	−.(1)41647 89943	.(3)46958 80252	14
15	0.62707 92809	−.(1)35883 40140	.(3)37500 17543	15
16	0.58502 29532	−.(1)31239 29587	.(3)30421 80906	16
17	0.54826 66002	−.(1)27442 67469	.(3)25019 32673	17
18	0.51586 56594	−.(1)24299 01253	.(3)20824 49142	18
19	0.48708 78486	−.(1)21666 60247	.(3)17517 73888	19
20	0.46135 66062	−.(1)19440 22295	.(3)14875 87453	20

p	D_5	E_5	F_5	p
3	1.27407 40741	− 0.11851 85185	.(1)11851 85185	3
4	0.31746 03175	−.(1)19047 61905	.(2)12698 41270	4
5	0.12768 95944	−.(2)56437 38977	(3)28218 69488	5
6	.(1)64197 53086	−.(2)22446 68911	.(4)89786 75645	6
7	.(1)36852 03685	−.(2)10656 01066	.(4)35520 03552	7
8	.(1)23117 62312	−.(3)56980 05698	.(4)16280 01628	8
9	.(1)15458 03507	−.(3)33198 46457	.(5)82996 16142	9
10	.(1)10847 37988	−.(3)20639 83488	.(5)45866 29974	10
11	.(2)79052 85098	−.(3)13501 76788	.(5)27003 53577	11
12	.(2)59393 10287	−.(4)92017 48332	.(5)16730 45152	12
13	.(2)45755 21097	−.(4)64862 67356	.(5)10810 44559	13
14	.(2)35996 92489	−.(4)47031 15814	.(6)72355 62791	14
15	.(2)28829 97519	−.(4)34930 30313	.(6)49900 43304	15
16	.(2)23447 56961	−.(4)26484 45363	.(6)35312 60483	16
17	.(2)19326 72711	−.(4)20444 80223	.(6)25556 00279	17
18	.(2)16118 23181	−.(4)16033 34937	.(6)18862 76396	18
19	.(2)13582 79996	−.(4)12750 47715	.(6)14167 19683	19
20	.(2)11552 71331	−.(4)10266 67694	.(6)10807 02835	20

p	A_6	B_6	C_6	p
3	31.86666 66667	− 2.95555 55556	1.24444 44444	3
4	9.86349 20646	− 1.68888 88889	0.31111 11111	4
5	6.24338 62423	− 1.20740 74074	0.14814 81481	5
6	4.87864 28716	− 0.94276 09428	.(1)87542 08754	6
7	4.17145 81715	− 0.77415 17742	.(1)58016 05802	7
8	3.75275 83528	− 0.65703 18570	.(1)41336 44134	8
9	3.47631 97577	− 0.57083 96179	.(1)30970 33685	9
10	3.28104 57516	− 0.50471 27623	.(1)24079 80736	10
11	3.13617 37676	− 0.45235 63811	.(1)19263 84589	11
12	3.02462 67843	− 0.40986 52428	.(1)15764 04780	12
13	2.93620 42788	− 0.37468 59903	.(1)13140 09662	13
14	2.86445 51800	− 0.34507 78315	.(1)11121 84648	14
15	2.80510 85568	− 0.31981 26863	.(2)95359 72754	15
16	2.75522 87178	− 0.29799 91485	.(2)82670 73153	16
17	2.71273 35648	− 0.27897 43935	.(2)72358 73754	17
18	2.67610 66890	− 0.26223 52558	.(2)63864 45096	18
19	2.64421 82442	− 0.24739 28474	.(2)56784 05678	19
20	2.61620 96162	− 0.23414 18341	.(2)50820 05082	20

p	D_6	E_6	F_6	p
3	− 0.88888 88889	10.85234 56791	− 3.56543 20985	3
4	−.(1)12698 41270	28.89171 07594	− 5.51675 48502	4
5	−.(2)42328 04233	1.42790 35865	− 0.19461 49324	5
6	−.(2)19240 01924	0.86407 78152	−.(1)91881 78077	6
7	−.(2)10360 01036	0.58217 60133	−.(1)50816 67304	7
8	−.(3)62160 06216	0.41986 26065	−.(1)31099 76443	8
9	−.(3)40221 21669	0.31750 86255	−.(1)20431 95246	9
10	−.(3)27519 77984	0.24868 72810	−.(1)14149 24384	10
11	−.(3)19656 98560	0.20013 83997	−.(1)10205 88045	11
12	−.(3)14529 07631	0.16458 67560	−.(2)76047 24004	12
13	−.(3)11042 09800	0.13776 06312	−.(2)58190 22630	13
14	−.(4)85882 98444	0.11701 40143	−.(2)45522 17628	14
15	−.(4)68114 09110	0.10063 52146	−.(2)36284 76722	15
16	−.(4)54930 71863	.(1)87476 39340	−.(2)29389 67830	16
17	−.(4)44943 31524	.(1)76744 10943	−.(2)24138 37900	17
18	−.(4)37238 74691	.(1)67875 46964	−.(2)20068 35826	18
19	−.(4)31200 03120	.(1)60462 05713	−.(2)16865 01032	19
20	−.(4)26400 02640	.(1)54201 64924	−.(2)14309 22600	20

p	G_6	H_6	I_6	p
3	0.26864 19754	1 26419 75310	–.(1)98765 43210	3
4	0.24409 17107	1.19223 98589	–.(1)56437 38977	4
5	.(2)60764 25632	(1)30570 25279	–.(2)10346 85479	5
6	.(2)22147 39993	.(1)11372 98915	– (3)29928 91882	6
7	.(3)99719 21083	.(2)51964 49641	– (3)11182 23340	7
8	.(3)51454 71812	.(2)27108 69377	–.(4)49333 38267	8
9	.(3)29218 90503	.(2)15524 53840	–.(4)24473 22709	9
10	.(3)17816 50932	.(3)95299 97834	–.(4)13250 26436	10
11	.(3)11479 15011	.(3)61737 49551	–.(5)76774 75857	11
12	.(4)77282 94532	.(3)41752 74961	–.(5)46962 67091	12
13	.(4)53932 11190	.(3)29248 26113	–.(5)30029 01554	13
14	.(4)38778 42203	.(3)21098 48164	–.(5)19924 01348	14
15	.(4)28596 27483	.(3)15602 20206	–.(5)13639 45170	15
16	.(4)21549 40811	.(3)11786 12681	– (6)95910 77856	16
17	.(4)16546 77743	.(4)90694 43517	–.(6)69030 58224	17
18	.(4)12915 92277	.(4)70928 04898	–.(6)50706 35472	18
19	.(4)10229 00232	.(4)56268 09892	–.(6)37922 29454	19
20	.(5)82061 36861	.(4)45209 40193	–.(6)28818 74227	20

p	J_6
3	(2)79012 34568
4	.(2)28218 69488
5	.(4)37624 92651
6	.(5)85511 19662
7	.(5)26311 13742
8	.(6)98666 76533
9	.(6)42562 13407
10	.(6)20385 02210
11	.(6)10589 62187
12	(7)58703 33864
13	.(7)34318 87490
14	.(7)20972 64577
15	.(7)13306 78214
16	.(8)87191 61687
17	.(8)58749 43169
18	.(8)40565 08378
19	(8)28620 59966
20	.(8)20584 81591

TABLE II

$$C_x = C_{-x} = \frac{(2p)!}{2^{2p}(p-x)!(p+x)!}$$

(The number in parenthesis indicates the number of ciphers between the decimal point and the first significant figure.)

x	$p=3$	$p=4$	$p=5$	$p=6$
0	0.31250 00000	0.27343 75000	0.24609 37500	0 22558 59375
1	0.23437 50000	0.21875 00000	0.20507 81250	0.19335 93750
2	0.09375 00000	0.10937 50000	0.11718 75000	0.12084 96094
3	0.01562 50000	0.03125 00000	0.04394 53125	0.05371 09375
4		0.00390 62500	.(2)97656 25000	.(1)16113 28125
5			.(3)97656 25000	.(2)29296 87500
6				.(3)24414 06250

x	$p=7$	$p=8$	$p=9$	$p=10$
0	0.20947 26563	0.19638 06152	0.18547 05810	0.17619 70520
1	0.18328 85742	0.17456 05469	0.16692 35229	0.16017 91382
2	0.12219 23828	0.12219 23828	0.12139 89258	0.12013 43536
3	.(1)61096 19141	.(1)66650 39062	.(1)70816 04003	.(1)73928 83301
4	.(1)22216 79688	.(1)27770 99609	.(1)32684 32617	.(1)36964 41650
5	.(2)55541 99219	.(2)85449 21875	.(1)11672 97363	.(1)14785 76660
6	.(3)85449 21875	.(2)18310 54687	.(2)31127 92968	.(2)46205 52063
7	.(4)61035 15625	.(3)24414 06250	.(3)58364 86816	..(2)10871 88721
8		.(4)15258 78906	.(4)68664 55078	.(3)18119 81201
9			.(5)38146 97265	.(4)19073 48633
10				.(6)95367 43164

x	$p=11$	$p=12$	$p=13$	$p=14$
0	0.16818 80951	0.16118 02578	0.15498 10171	0.14944 59808
1	0.15417 24205	0.14878 17764	0.14391 09445	0.13948 29154
2	0.11859 41696	0.11689 99672	0.11512 87556	0.11332 98688
3	.(1)76239 10904	.(1)77933 31146	.(1)79151 01945	.(1)79997 55442
4	.(1)40660 85815	.(1)43837 48770	.(1)46559 42321	.(1)48887 39437
5	.(1)17789 12544	.(1)20629 40598	.(1)23279 71160	.(1)25730 20756
6	.(2)62785 14862	.(2)80225 46768	.(2)98019 83833	.(1)11578 59340
7	.(2)17440 31906	.(2)25334 35822	.(2)34306 94342	.(2)44108 92725
8	.(3)36716 46118	.(3)63335 89554	.(3)98019 83833	.(2)14034 65867
9	.(4)55074 69177	.(3)12063 98010	.(3)22277 23598	.(3)36612 15305
10	.(5)52452 08740	.(4)16450 88196	.(4)38743 01910	.(4)76275 31886
11	.(6)23841 85791	.(5)14305 11475	.(5)48428 77388	.(4)12204 05102
12		.(7)59604 64477	.(6)38743 01910	.(5)14081 59733
13			.(7)14901 16119	.(6)10430 81284
14				.(8)37252 90298

x	$\rho=15$	$\rho=16$	$\rho=17$
0	0.14446 44481	0.13994 99341	0.13583 37596
1	0.13543 54201	0.13171 75850	0 12828 74396
2	0 11153 50518	0.10976 46542	0.10803 15281
3	.(1)80553 09299	.(1)80879 21888	(1)81023 64605
4	(1)50875 63768	.(1)52571 49227	.(1)54015 76403
5	.(1)27981 60072	(1)30040 85273	(1)31919 40602
6	(1)13324 57177	.(1)15020 42636	(1)16653 08140
7	.(2)54509 61180	.(2)65306 20158	(2)76326 62309
8	(2)18959 86497	(2)24489 82559	(2)30530 64924
9	.(3)55299 60617	(3)78367 44189	(2)10568 30166
10	(3)13271 90548	.(3)21098 92666	.(3)31313 48640
11	(4)25522 89516	(4)46886 50370	(4)78283 71599
12	(5)37811 69653	.(5)83725 89946	.(4)16196 63089
13	(6)40512 53200	.(5)11548 39992	.(5)26994 38482
14	.(7)27939 67724	(6)11548 39992	.(6)34831 46429
15	.(9)93132 25746	(8)74505 80597	.(7)32654 49777
16		(9)23283 06436	.(8)19790 60471
17			.(10)58207 66091

x	$\rho=18$	$\rho=19$	$\rho=20$
0	0.13206 05996	0 12858 53206	0.12537 06876
1	0.12511 00417	0.12215 60546	0.11940 06549
2	0.10634 35354	0.10470 51897	0.10311 87474
3	.(1)81023 64606	(1)80908 55565	.(1)80701 62839
4	.(1)55243 39504	.(1)56284 21262	(1)57163 65345
5	.(1)33626 41437	.(1)35177 63289	(1)36584 73821
6	.(1)18214 30779	.(1)19699 47442	(1)21106 57973
7	(2)87428 67737	.(2)98497 37209	.(1)10944 15245
8	.(2)36989 05581	.(2)43776 60982	(2)50812 13640
9	.(2)13699 65030	.(2)17197 95385	.(2)21025 71161
10	.(3)44034 59025	.(3)59303 28916	.(3)77094 27591
11	.(3)12147 47317	.(3)17790 98675	.(3)24869 12126
12	.(4)28344 10407	.(4)45912 22386	.(4)69944 40355
13	(5)54859 55626	.(4)10043 29897	.(4)16956 21904
14	.(6)85718 05666	.(5)18260 54358	.(5)34909 86273
15	.(6)10390 06747	(6)26853 74056	.(6)59845 47897
16	(8)91677 06595	(7)30689 98921	.(7)83118 72080
17	(9)52386 89483	.(8)25574 99101	.(8)89858 07654
18	(10)14551 91523	.(9)13824 31946	.(9)70940 58673
19		(11)36379 78808	.(10)36379 78807
20			(12)90949 47018

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APPROXIMATION AND GRADUATION ACCORDING TO THE PRINCIPLE OF LEAST SQUARES BY ORTHOGONAL POLYNOMIALS*

By

CHARLES JORDAN, *Dr. Sc. University of Budapest*

PREFACE

In the present paper the mathematical theory of approximation by orthogonal polynomials is given in its entirety and accompanied by nearly all of the necessary demonstrations. This has necessitated some mathematical preliminaries.

Statisticians less interested in mathematics will find it sufficient to read § 12 on the recapitulation of the operations, the beginning of § 9 concerning the computation of the mean binomial moments and the mean orthogonal moments, § 11 dealing with the method of addition of differences, and the examples of § 13. In paragraphs § 14 and § 15 dealing with correlation and in § 17 treating graduation, one may observe the end-formulae and skip the rest. With the aid of these sections and the tables, one may readily employ the methods and attain results with a very small amount of labor.

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Bibliographical and Historical Notes.

§ 18. Chebisheff.
§ 19. Gram.
§ 20. Jordan.
§ 21. Essher.
§ 22. Lorentz.

§ 1. *Introduction.* It has been shown that in any case of approximating a function $F(x)$ it is advantageous to develop that function in a series of orthogonal functions. It was *Fourier* who first used such an expansion in his treatment of trigonometric series. The first expansion in orthogonal polynomials was performed by *Legendre*. In Legendre's polynomials the variable x is a continuous one, it takes on every value between -1 and $+1$. Orthogonal polynomials with respect to a discontinuous variable, where x assumes only the N values $x_0, x_1, \cdots, x_{N-1}$ have been deduced by *Chebisheff*[1] who has treated the particular case of two orthogonal polynomials with respect to equidistant variables.

Since then several authors have investigated this subject. *Poin-*

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[1] Chebisheff. Sur les fractions continues, Journal de Mathématiques pures and appliquées 1858, T. III (Oeuvres Tome I. 203).
Sur l'interpolation par la méthode des moindres carrés. Mem. Acad. Imp. de St. Pétersbourg, 1859 (Oeuvres Tome I. p. 473).
Sur l'interpolation des valeurs équidistantes, 1875 (Oeuvres Tome II. p. 219).

caré[2] and *Quiquet*[3] considered orthogonal polynomials of a discontinuous variable in the case of non-equidistant values. *Gram*[4] employed these polynomials for $x = -n, \cdots, -1, 0, 1, \cdots, n$. *Jordan*[5] considered the general case of equidistant orthogonal polynomials for equidistant values of x between a and b, the interval of two consecutive values of x being h , moreover he treated the particular case of polynomials relative to $x = 0, 1, 2, \cdots, (n-1)$ which Chebisheff has examined in another way *Essher* in his first publication[6] used such polynomials with respect to $x = -\frac{1}{2}(N-1) \cdots \frac{1}{2}(N-1)$, and in his second[7] for $x = -n, \cdot, 0, 1, \cdots, n$. *Lorentz*[8] also introduced orthogonal polynomials for $x = -n, \cdots, 0, 1, \cdots, n$ and for $x = -2n+1, \cdots, -1, 1, 3, \cdots, 2n-1$, the interval in this latter case being obviously equal to two

In later publications I showed new methods for using orthogonal polynomials for approximation[9] and graduation[10] which permit the results to be reached very rapidly. In the present paper the general case of orthogonal polynomials for equidistant values of the variable is to be discussed; the formulae given are valid for all orthogonal polynomials of equidistant values such as the polynomials of Gram, Essher and Lorentz. These are also discussed in this paper. At the end some very useful tables are appended.

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[2] ***Poincairé.*** Calcul des probabilités, Paris, 1896. p 251.

[3] ***A. Quiquet.*** Sur une methode d'interpolation exposée par Henri Poincare. Proc of the fifth International Congress of Mathematicians. Cambridge, 1913 p. 385.

[4] *J. Gram.* Ueber partielle Ausgleichung mittelst Orthogonalfunktionen, Bull. de l'Association des Actuaires Suisses, 1915.

[5] *Ch. Jordan.* Sur une série de Polynomes dont chaque somme partielle represente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés Proc. of the London Math. Soc. 1921.

[6] *F. Essher.* Ueber die Sterblichkeit in Schweden, Lund 1920

[7] *F Essher* On some methods of Interpolation. Scandia, 1930.

[8] *P. Lorentz* Der Trend, Vierteljahreshefte zur Konjunkturforschung, Berlin 1928. Zweite Auflage, 1931.

[9] *K. Jordan,* Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate, Mitt. der Ungarischen Landeskommission fur Wirtschaftsstatistik und Konjuncturenforschung 1930.

Ch. Jordan, Sur la détermination de la tendance séculare des grandeurs statistiques par la méthode des moindres carrés. Journal de la Société Hongroise de Statistique, 1929.

[10] *Ch Jordan,* Statistique Mathématique, p. 291, Paris 1927, Gauthier Villars.

§ 2. *Some formulae of the Calculus of Finite Differences.* Since the functions considered in this paper correspond to $x=a$, $a+h$, $a+2h$, $a+3h\cdot$, the increment h being constant, the Calculus of Finite Differences will be found very useful.

By the first difference of $F(x)$ we mean $F(x+h)-F(x)$ which is denoted by $\Delta F(x)$, so that $\Delta F(x)=F(x+h)-F(x)$.

The n-th difference of $F(x)$ will be defined as

$$\Delta^n F(x)=\Delta\left[\Delta^{n-1}F(x)\right].$$

We shall term $F(x)$ the *indefinite sum of* $f(x)$ and denote it by $\Sigma f(x)$ if $\Delta F(x)=f(x)$. It follows that $\Sigma f(x)=F(x)+C$, where C is an arbitrary constant or a periodic function of periodicity h.

The n-th sum of $F(x)$ will be defined by

$$\Sigma^n f(x)=\Sigma\left[\Sigma^{n-1}f(x)\right].$$

This contains an arbitrary polynomial of the $(n-1)th$ degree, considering only the polynomial and neglecting the arbitrary periodic function.

It would be more precise to add the increment h to the above notation for the difference Δ and the indefinite sum Σ. Thus we could use $\underset{h}{\Delta}$ and $\underset{h}{\Sigma}$, respectively; but since in our formulae the increment will generally equal h we shall omit this index except in cases where doubt might otherwise arise.

By the *definite sum of* $f(x)$ *between* a *and* b, the following sum is understood (b being equal to $a+nb$ and n an integer)

$$f(a)+f(a+h)+f(a+2h)+\cdots\cdot+f(b-h)$$

and is denoted by

$$\sum_{x=a}^{b} f(x).$$

It can be shown that if $F(x)$ is the definite sum of $f(x)$, the above sum is equal to the difference of the values of $F(x)$ taken at the limits, so that we have

$$\sum_{x=a}^{b} f(x) = f(a) + f(a+h) + \cdots + f(b-h) = F(b) - F(a).$$

According to our definition it is evident that the value of the function $f(x)$ at the upper limit, i e. $f(b)$, is not included in the sum between a and b This terminology, although rather unusual, will be adhered to throughout this paper.

We shall have occasion to employ the following formulae of the calculus of finite differences.

Formula of differencing by parts, or of a product·

$$(1)\quad \Delta\left[u(x)\cdot v(x)\right] = u(x)\,\Delta v(x) + v(x+h)\cdot\Delta\, u(x)$$

Formula of the n-th difference of a product·

$$(2)\quad \Delta^{n}\left[u(x)\cdot v(x)\right] = \sum_{s=0}^{n+1} \binom{n}{s} \Delta^{n-s} u(x+sh)\cdot\Delta^{s} v(x).$$

Formula of the summation by parts, or of the sum of a product:

$$\sum\left[u(x)\; v(x)\right] = u(x)\cdot\sum v(x) - \sum\left[\Delta\, u(x)\cdot\sum v(x+h)\right].$$

Formula of the sum of a product, $u(x)$ being a polynomial of the n-th degree.

$$(3)\quad \begin{aligned} \sum\left[u(x)\cdot v(x)\right] &= u(x)\,\sum v(x) - \Delta u(x)\cdot\sum^{2} v(x+h) \\ &+ \Delta^{2} u(x)\cdot\sum^{3} v(x+2h) - \cdots + (-1)^{n}\Delta^{n} u(x)\cdot\sum^{n+1} v(x+nh). \end{aligned}$$

In the second member of this equation $\sum v(x)$ contains one arbitrary constant, $\sum^{2} v(x+h)$ contains, besides this, one more, and so on. Ultimately the second member will contain $n+1$ arbitrary constants — but since the first member contains but one constant it is evident that after simplification all of the terms of the arbitrary polynomial must necessarily vanish except the single constant term arising from $\sum^{n+1} v(x+nh)$.

Generalized binomial coefficients. We shall denote the generalized binomial coefficient of the n-th degree by

$$\binom{x}{n}_{h} = \frac{x(x-h)\cdot(x-2h)\cdots\cdot(x-nh+h)}{1\cdot 2\cdot 3\cdot\cdots\cdot n},$$

where the index h is associated with the decrement. If $h=1$ the index will be omitted, and the expression above will be equal to the ordinary binomial coefficient, $\binom{x}{n}$.

Let us mention the following well-known formulae:

$$\Delta^s \binom{x}{n}_h = h^s \binom{x}{n-s}$$

$$\Sigma \binom{x}{n}_h = \frac{1}{h} \binom{x}{n+1}.$$

Expansion of a function in a series of generalized binomial coefficients:

$$(4) \qquad f(x) = f(a) + \binom{x-a}{1}_h \frac{\Delta f(a)}{h} + \dots + \binom{x-a}{n} \frac{\Delta^n f(a)}{h^n}.$$

The generalized binomial coefficient can be expressed as an ordinary one by merely changing the variable. Thus if we place $(x-a)/h = \xi$ we have that

$$\binom{x}{n}_h = h \binom{\xi}{n}$$

and consequently if we write $F(\xi) = f(a+h\xi)$ it follows that $\Delta^s F(0) = \underset{h}{\Delta^s} f(a)$, and formula (4) may be written

$$(4') \qquad \begin{aligned} F(\xi) = F(0) + \binom{\xi}{1} \cdot \Delta F(0) + \binom{\xi}{2} \cdot \Delta^2 F(0) + \\ \dots + \binom{\xi}{n} \Delta^n F(0). \end{aligned}$$

Formulae (4) and (4′) are two different forms of Newton's series. The great importance of Newton's formula to the statistician is not yet sufficiently recognized by the latter, since he nearly always develops his functions in power series in spite of the fact that he is generally primarily concerned with the differences and the sums of his function. Now if a function be expanded in a Newton series as in (4) above, its m-th difference and its sum can be obtained immediately by means of the following formulae:

$$(5)\qquad \Delta^m f(x) = \Delta^m f(a) + \binom{x-a}{1}_h \frac{\Delta^{m+1} f(a)}{h} + \cdots\cdots + \binom{x-a}{n-m}_h \frac{\Delta^n f(a)}{h^{n-m}};$$

$$(6)\qquad \Sigma f(x) = \binom{x-a}{1}_h \frac{f(a)}{h} + \binom{x-a}{2}_h \frac{\Delta f(a)}{h^2} + \cdots\cdots + \binom{x-a}{n+1}_h \frac{\Delta^n f(a)}{h^{n+1}}$$

These operations would be very complicated if $f(x)$ were expanded into a power series. Although it is true that a power series would be more advantageous for determining either the derivatives or the integral of $f(x)$, we may remark that the statistician hardly ever needs these quantities. And for nearly all other operations, Newton's formula is at least as convenient as an expansion in a power series.

To illustrate the last remark — if $f(x)$ corresponding to a given value of x is needed, then in the case of a power series it is necessary to compute the values of $x, x^2, x^3, \cdots\cdot$ and these are obtained most readily by means of successive multiplication. In the case of a Newton series it is necessary to calculate $(x-a), (x-a)(x-a-h), (x-a)(x-a-h)(x-a-2h), \cdots\cdots\cdot$ and these should also be obtained by successive multiplication. The formula for changing the origin and that for changing the interval of observation are given in the following paragraph and are as simple as those arising in the case of power series. If a statistician expands a function into a power series and needing the differences of the function for $x = a$ calculates them separately, he doubles his work since these differences are precisely the coefficients in Newton's formula. In statistical research Newton's series should always be preferred to the power series.

§ 3. *Changing the origin* In mathematical statistics it frequently occurs that it is necessary to change the origin of a set of observations. For instance, if $f(a)$ is the value of some quantity corresponding to the first of January of the year 1901

$$a = \text{January } 1, 1901,$$

h represents the interval of a year, and it follows that $x = a + \xi h$ represents January 1, $(1901 + \xi)$ where ξ is an integer. If we know the *Newton* expansion of $f(x)$ in generalized binomial coefficients $\binom{x-a}{\nu}_h$ that is

$$f(x) = f(a) + \binom{x-a}{1}_h \frac{\Delta f(a)}{h} + \dots + \binom{x-a}{n}_h \frac{\Delta^n f(a)}{h^n},$$

and desire the values of $f(x)$ counted from the first of July of 1901, then denoting

$$c = \text{July } 1, 1901$$

we need $f(x)$ expanded into a series of generalized binomial coefficients $\binom{x-c}{\nu}_h$, that is

$$(\alpha) \qquad f(x) = f(c) + \binom{x-c}{1}_h \frac{\Delta f(c)}{h} + \dots + \binom{x-c}{n}_h \frac{\Delta^n f(c)}{h^n}.$$

The coefficients of this expansion must be so determined that $x = c + \xi h$ will correspond to July 1, $(1901 + \xi)$. These are easily obtained by putting $x = c$ in the first equation of this paragraph,

$$f(c) = f(a) + \binom{c-a}{1}_h \frac{\Delta f(a)}{h} + \dots + \binom{c-a}{n}_h \frac{\Delta^n f(a)}{h^n}$$

and also placing $x = c$ in the s-th difference of the same equation, so that we have

$$(7) \quad \Delta^s f(c) = \Delta^s f(a) + \binom{c-a}{1}_h \frac{\Delta^{s+1} f(a)}{h} + \dots + \binom{c-a}{n-s}_h \frac{\Delta^n f(a)}{h^{n-s}}.$$

Substituting these values into equation (α) yields the required expansion.

Remark. If $c-a=mh$ where m is an integer, from the above equations it follows that

$$f(c)=f(a+mh) \text{ and } \Delta^s f(c)=\Delta^s f(a+mh).$$

An example of this is given in § 13.

§ 4. *Changing the interval* h. Sometimes a changing of the interval is needed in Newton's formula, this occurs for instance in statistics when a function $f(x)$, giving the value of some quantity corresponding to the middle of the year x, is known for several consecutive years $(h=1)$ by its Newton expansion (4), and it is necessary to obtain the values of the quantity corresponding to the first of each month. It would of course be possible to calculate these values by placing $x=1/12, 2/12, 3/12, \cdots$ in Newton's formula, but it is more advantageous to change the interval $(h=1)$ in formula (4) and deduce another one in which both the increment of the differences and the decrement of the generalized binomial coefficients are $k=1/12$. The formula thus obtained leads, by the method of summation of the differences (§ 11), more rapidly to the results So, starting from formula

$$(8) \qquad f(x)=\sum_{m=0}^{n+1}\frac{1}{h^m}\binom{x-a}{m}_h \underset{h}{\Delta}^m f(a)$$

we have to deduce

$$(9) \qquad f(x)=\sum_{m=0}^{n+1}\frac{1}{k^m}\binom{x-a}{m}_k \underset{k}{\Delta}^m f(a).$$

To obtain this, it is sufficient to know that for $x=a$ the differences of the generalized binomial coefficients $\binom{x-a}{s}_h$ in a system of differences in which the increment is k may be written

$$(10) \qquad \left[\Delta_k^s \binom{x-a}{m}_h \right]_{x=a} = A_m^s .$$

To obtain these numbers, A_m^s, let us write the following identity,

$$\Delta_k^s \binom{x-a}{m}_h = \Delta_k^s \left[\binom{x-a}{m-1}_h \cdot \frac{x-a-mh+h}{m} \right]$$

and then deduce the s-th difference. By formula (2) we have

$$\Delta_k^s \binom{x-a}{m}_h = \frac{x-a-mh+h}{m} \Delta_k^s \binom{x-a}{m-1}_h + \frac{sk}{m} \Delta_k^{s-1} \binom{x-a}{m-1}_h .$$

Placing in this equation $x=a$, we obtain

$$(11) \qquad A_m^s = \frac{sk-(m-1)h}{m} A_{m-1}^s + \frac{sk}{m} A_{m-1}^{s-1} .$$

The complete solution of this *Equation of Partial Differences* with the interval k would be very complicated, but one may readily solve it for some particular values of s and then deduce successively the other values of A_m^s starting from the initial values which follow immediately from (10). These are that

$$A_1^1 = k,$$

$$A_m^0 = 0, \text{ except that } A_0^0 = 1,$$

and $A_m^{m+\nu} = 0$

Equation (11) can be solved *first* for $s=m$ yielding

$$A_m^m = kA_{m-1}^{m-1} = k^{m-1} A_1^1 = k^m ,$$

and *secondly* for $s=1$ from which we obtain

$$A_m^1 = \frac{k-(m-1)h}{m} A_{m-1}^1$$

$$A_{m-1}^1 = \frac{k-(m-2)h}{m} A_{m-2}^1$$

. . . .

so that by multiplying we easily obtain

$$A_m^1 = \binom{k}{m}_h ,$$

and *thirdly* for $s=m-1$ we may express successively the values of

$$A_m^{m-1}, A_{m-1}^{m-2}, \ldots\ldots, A_2^1$$

and then multiply each $A_{m-\nu}^{m-\nu-1}$ by $k^\nu(m-\nu)/m$ and adding the products obtain the result

$$A_m^{m-1} = (m-1)k^{m-2}\cdot\binom{k}{2}_h.$$

The other values of A_m^s are obtained as indicated above. The following table contains the numbers necessary for binomial coefficients up to the fifth degree.

$$A_1^1 = k \qquad A_2^1 = \binom{k}{2}_h \qquad A_2^2 = k^2$$

$$A_3^1 = \binom{k}{3}_h \qquad A_3^2 = 2k\binom{k}{2}_h \qquad A_3^3 = k^3$$

$$A_4^1\binom{k}{4}_h \qquad A_4^2 = \frac{k}{6}(7k-11h)\cdot\binom{k}{2}_h \qquad A_4^3 = 3k^2\binom{k}{2}_h \qquad A_4^4 = k^4$$

$$A_5^1 = \binom{k}{5}_h \qquad A_5^2 = \frac{k}{2}(3k-5h)\cdot\binom{k}{3}_h \qquad A_5^3 = \frac{k^2}{2}(5k-7h)\binom{k}{2}_h$$

$$A_5^4 = 4k^3\binom{k}{2}_h \qquad A_5^5 = k^5$$

The numbers A_m^s being known, we can immediately express $\binom{x-a}{m}_h$ in a *Newton* series, with the increment equal to k,

$$\binom{x-a}{m}_h = \binom{x-a}{1}_k \frac{A_m^1}{k} + \binom{x-a}{2}_k \frac{A_m^2}{k^2} + \binom{x-a}{3}_k \frac{A_m^3}{k^3} + \cdots .$$

It follows from (8) that

$$f(x) = \sum_{m=0}^{n+1} \Delta_h^m f(a) \cdot \frac{1}{h^m} \sum_{\nu=0}^{m+1} \binom{x-a}{\nu}_k \cdot \frac{A_m^\nu}{k^\nu}$$

and consequently the s-th difference of $f(x)$ for $x=a$ will be

$$\Delta_k^s f(a) = \sum_{m=s}^{n+1} \frac{1}{h^m} \cdot A_m^s \Delta_h^m f(a) .$$

When these values are placed in equation (9) the problem is solved. An example is given in § 13.

§ 5. *The problem of approximation.* The number of the given values will always be denoted by N in this paper. The values $y_0, y_1, y_2, \cdots, y_{N-1}$ correspond to $x=a, a+h, a+2h, \cdots, b-h$ where $b=a+Nh$. A parabola of the n-th degree, $y=f_n(x)$ is to be determined according to the principle of least squares, that is, so that the sum of the squares of the deviations $\left[f_n(x)-y\right]$ for $x=0, 1, 2, \cdots, N-1$ shall be a minimum. Hence the parameters in $f_n(x)$ must be so determined that the expression

$$S = \sum_{x=a}^{b} \left[f_n(x)-y\right]^2 \tag{12}$$

shall be a minimum.

To solve the problem in the ordinary way would require the solution of $n+1$ determinants of the n-*th* order. This would be very laborious, as those who have employed *Gauss's* method to solve this problem know. It is far more convenient to first expand the function $f(x)$ into a series of *orthogonal polynomials*

Let $U_\nu = U_\nu(x)$ be such a polynomial — it is termed orthogonal if it satisfies the following relation

$$(13) \qquad \sum_{x=a}^{b} U_\nu U_\mu = 0$$

for all values of ν different from μ. The expansion of $f_n(x)$ can be written as follows

$$(14) \qquad f_n(x) = a_0 U_0 + a_1 U_1 + a_2 U_2 + \cdots + a_n U_n$$

where the a_ν are constant parameters which must be evaluated according to the principle of least squares.

To render expression (12) a minimum it is necessary that the first derivative of S with respect to a_ν should vanish for all values of ν. This will produce $n+1$ equations which determine the $n+1$ parameters, namely

$$\sum_{x=a}^{b} U_\nu \left[a_0 U_0 + a_1 U_1 + a_2 U_2 + \cdots + a_n U_n - y \right] = 0$$

As a consequence of relation (13) these equations are so simplified that we may write

$$(15) \qquad a_\nu \sum_{x=a}^{b} U_\nu^2 - \sum_{x=a}^{b} U_\nu \cdot y = 0.$$

The second condition of a minimum, namely that the expression

$$\sum_{\nu=0}^{n+1} \sum_{\mu=0}^{n+1} \frac{\partial^2 S}{\partial a_\nu \cdot \partial a_\mu} \, da_\nu \, da_\mu$$

shall be a *positive definite form* for all values of da_ν and da_μ, is also satisfied since in consequence of (13) this quantity is equal to

$$\sum_{x=a}^{b} U_\nu^2 (da_\nu)^2 > 0.$$

From (15) it may be concluded that the coefficients a_ν are independent of the degree n of the parabola of approximation. Consequently, if the coefficients $a_0, a_1, \cdots, a_n$ corresponding to a parabola of degree n have been calculated, then to obtain a further parabola of degree $n+1$ it is sufficient to determine only one further coefficient, a_{n+1} — the others will remain unchanged. This is of great importance. If the series (14) is limited at any term, the remaining expression will always satisfy condition (12).

§ 6. *Deduction of the polynomial* U_ν. Instead of starting from relation (13) we shall employ the following equivalent formula,

$$\sum_{x=a}^{b} F_{m-1}(x) \cdot U_m(x) = 0,$$

where $F_{m-1}(x)$ is an arbitrary polynomial of degree $m-1$. If we were to expand $F_{m-1}(x)$ into a series of U_ν polynomials we would return back to condition (13) again.

Applying formula (3), of the sum of a product, to the above expression yields

$$\Sigma\left[F_{m-1} \cdot U_m\right] = F_{m-1} \Sigma U_m(x) - \Delta F_{m-1} \cdot \Sigma^2 U_m(x+h)$$
$$+ \Delta^2 F_{m-1} \cdot \Sigma^3 U_m(x+2h) - \cdots + (-1)^{m-1} \cdot \Delta^{m-1} F_{m-1} \Sigma^m U_m(x+mh-h).$$

Now, $\Sigma U_m(x)$ contains an arbitrary constant to which may be assigned such a value that $\Sigma U_m(a)=0$. But $\Sigma^2 U_m(x+h)$ contains an additional constant which can be chosen so that $\Sigma^2 U_m(a+h)=0$. Continuing after this fashion we may dispose of all these arbitrary constants in such a way that the expression for the definite sum will vanish for the lower limit $x=a$, that is

$$\Sigma\left[F_{m-1}\cdot U_m\right]_{x=a}=0\ .$$

But in order that the definite sum may be equal to zero it is necessary for the above expression to vanish also for the upper limit, $x=b$. But since $F(b)$ is arbitrary for all values of s it follows each expression obtained for $s=0,1,2,\ ,m-1$ must vanish separately for $x=b$. From this we conclude that $(x-a)$ and $(x-b)$ must both be multiplying factors of $U(x)$. Considering for the moment only the first of these factors we may therefore write

$$U(x)=(x-a)\,\omega(x).$$

Applying to this expression the formula for the sum of a product, (3), we have

$$\Sigma^2 U_m(x)=\sum_{\nu=0}^{m+1}(-1)^\nu\frac{1}{h^{\nu+1}}\binom{x-a+\nu h}{\nu+2}_h\,\Delta^\nu\omega(x).$$

By successive summation we should find that $(x-a)(x-a-h)\cdots(x-a-mh+h)$ is a multiplying factor of $\Sigma^m U_m(x)$ and that we can assign the following form to this expression

$$\Sigma^m U_m(x)=\binom{x-a}{m}_h\,\psi(x)\ . \tag{16}$$

As $(x-b)$ must also be a multiplying factor of $\Sigma U_m(x)$, the same reasoning leads to the expression of $\Sigma^m U_m(x)$

$$\Sigma^m U_m(x) = C\binom{x-a}{m}_h \binom{x-b}{m}_h .$$

As this sum must be of degree $2m$, it follows that C is an arbitrary constant and we conclude that the general formula of the orthogonal polynomials with respect to $x=a, a+h, \cdot\cdot, b-h$, is the following

$$(17) \qquad U_m(x) = C\Delta^m \left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right].$$

Starting from this expression, there are two different ways of deducing the expansion of $U_m(x)$ in *Newton*-series, as has been shown in the paper [5]. First, we can utilize formula (2), giving the m-th difference of a product, and obtain

$$(18) \quad U_m(x) = Ch^m \sum_{s=0}^{m+1} \binom{m}{s}\binom{x-a+sh}{s}_h \binom{x-b}{m-s}_h$$

Secondly, we can develop $\Sigma^m U_m$ into a *Newton*-series of generalized binomial coefficients $\binom{x-b}{s}_h$. According to formula (5), we have then that

$$(19) \quad \Sigma^m U_m(x) = C\sum_{s=0}^{2m+1} \frac{1}{h^s}\binom{x-b}{s}_h \Delta^s \left[\binom{x-a}{m}_h \binom{x-b}{m}_h\right]_{x=b} .$$

The s-th difference in this formula can be written according to (2) in the following manner

$$\left[h^s\sum_{\nu=0}^{s+1}\binom{s}{\nu}\binom{x-a+\nu h}{m-s+\nu}_h\binom{x-b}{m-\nu}_h\right]=\binom{s}{m}\binom{x-a+mh}{2m-s}_h h^s,$$

so it follows that

$$(20)\quad \Sigma^m U_m(x) = C\sum_{s=m}^{2m+1}\binom{s}{m}\binom{b-a+mh}{2m-s}_h\binom{x-b}{s}_h;$$

and finally putting $s=m+\nu$ into this expression, and determining the m-th difference, we see that

$$(21)\quad U_m = Ch^m\sum_{\nu=0}^{m+1}\binom{x-b}{\nu}_h\binom{m+\nu}{m}\binom{b-a+mh}{m-\nu}_h.$$

Let us note that $U_0=C$.

As $\Sigma^m U_m(x)$ is symmetric with respect to a and b, we can get two other formulae for $U_m(x)$ from (18) and (21) changing a into b and inversely. For instance, remarking that $b-a=Nh$, and that

$$\binom{a-b-mh}{m-\nu}_h=\binom{-N+m}{m-\nu}h^{m-\nu}=(-1)^{m-\nu}\binom{N-\nu-1}{m-\nu}h^{m-\nu}$$

it follows from (21) that

$$(21')\quad U_m(x)=Ch^{2m}\sum_{\nu=0}^{m+1}(-1)^{m-\nu}h^{-\nu}\binom{m+\nu}{m}\binom{N-\nu-1}{m-\nu}\binom{x-a}{\nu}_h.$$

The constant C is arbitrary, whatever value may be chosen for it. The orthogonal polynomials introduced by different investigators differ only in the value attributed to C. As these polynomials are closely related to *Legendre's* polynomials it seems most advisable to choose C in such a manner that for $h=0$ the limit of the polynomial U_m shall be equal to *Legendre's* polynomial P_m. For this purpose, we must put into (19) and (21) $a=-1$ and, $b=1$ and, $C=1.2.3\dots m/2^m h^m$; then, deducing the limit for $h=0$, we obtain two known formulae for *Legendre's* polynomials.[11]

The choice of C is only important if we want to compute numerical values of $U_m(x)$ corresponding to any value of x; in this case C should be chosen so that the calculation of $U_m(x)$ shall be as short as possible. As we shall see later Essher in his first paper and also Gram proceeded in this manner. The above-given value that I chose for C is in this respect also very acceptable; and whenever numerical values of a_m are needed, we will adopt this value. It will be shown that our problem can be solved in a general way by leaving the constant C arbitrary.

§ 7. *Determination of the coefficients* a_m. These are given by formula (15); but first it is necessary to know the value of

$$\sum_{x=a}^{b} U_m^2$$

To determine this quantity we shall apply formula (3) of the sum of a product to the following indefinite sum

$$\begin{aligned}\Sigma\left[U_m U_m\right] = {} & U_m(x)\Sigma U_m(x) - \Delta U_m(x)\Sigma^2 U_m(x+h) \\ & + \Delta^2 U_m(x)\Sigma^3 U_m(x+2h) - \dots \\ & + (-1)^m \Delta^m U_m(x)\Sigma^{m+1} U_m(x+mh).\end{aligned} \tag{22}$$

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[11] I proceeded in this way in the paper [5] *loc. cit.*; formulae (19) and (21) of the present paper are identical with (9) and (13) of the first paper, only the notation is somewhat different.

Let us now determine the quantities $\Sigma^{\mu} U_m(x+\mu h-h)$; they are easily obtained, if $\mu < m+1$, by deducing the $(m-\mu)$-th difference of $\Sigma^m U_m(x+\mu h-h)$. Starting from formula (20), after having replaced x by $x+\mu h-h$, it follows that

$$\Delta^{m-\mu}\left[\Sigma^m U_m\right] = \Sigma^{\mu} U_m(x+\mu h-h) = \tag{23}$$

$$Ch^{\mu} \sum_{s=m}^{2m+1} \binom{x+\mu h-h-b}{s-m+\mu}\binom{s}{m}\binom{b-a+mh}{2m-s}.$$

At the upper limit, $x=b$ the first generalized binomial coefficient figuring in this formula can be expressed by an ordinary one,

$$\binom{\mu h-h}{s-m+\mu}_h = \binom{\mu-1}{s-m+\mu} h^{s-m+\mu}$$

but since $s \geqq m$, it follows that this expression is equal to zero; indeed an ordinary binomial coefficient $\binom{r}{t}$ is equal to zero, if r and t are integers, and if $r < t$.

$\Sigma^m U_m(x+\mu h-h)$ being symmetrical with respect to a and b, its $(m-\mu)$-th difference $\Sigma^{\mu} U_m(x+\mu h-h)$ will be symmetrical too; but this quantity, as we have seen, is equal to zero for $x=b$, therefore it must also be equal to zero for $x=a$.

We conclude that at both limits $x=a$ and $x=b$ all the terms of (22) will vanish in which $\mu < m+1$, — that is all terms except the last.

To evaluate this last term, we shall determine first the indefinite sum of $\Sigma^m U_m$. From (20) we easily get, after putting $x+mh$ instead of x,

$$\Sigma^{m+1} U_m(x+mh) = \frac{C}{h} \sum_{s=m}^{2m+1} \binom{x+mh-b}{s+1}_h \binom{s}{m} \binom{b-a+mh}{2m-s}_h .$$

Since $m < s+1$, this expression will vanish at the upper limit $x=b$ and its value at the lower limit, $x=a$, will be

$$\Sigma^{m+1} U_m(a+mh) = \frac{C}{h} \sum_{s=m}^{2m+1} \binom{s}{m} \binom{a-b+mh}{s+1}_h \binom{b-a+mh}{2m-s}_h .$$

Noting that $b-a=Nh$, we can express the two generalized binomial coefficients of this formula by ordinary ones; indeed they are equal to

$$\binom{-N+m}{s+1}\binom{N+m}{2m-s} h^{2m+1} = (-1)^{s+1} \binom{N-m+s}{s+1}\binom{N+m}{2m-s} h^{2m+1}$$

$$= (-1)^{s+1} \binom{N+m}{2m+1}\binom{2m+1}{s+1} h^{2m+1} ,$$

so that we have

$$\Sigma^{m+1} U_m(a+mh) = C h^{2m} \binom{N+m}{2m+1} \sum_{s=m}^{2m+1} (-1)^{s+1} \binom{s}{m}\binom{2m+1}{s+1} .$$

According to a known formula of *Combinatory Analysis* (*Netto*, p. 250.) the last sum of this equation is equal to $(-1)^{m+1}$. So it follows that

$$\Sigma^{m+1} U_m(a+mh) = (-1)^{m+1} C h^{2m} \binom{N+m}{2m+1} .$$

Moreover from (21) we deduce that

$$\Delta^m U_m(x) = C h^{2m} \binom{2m}{m}.$$

Now we have determined the values, at the limits, of all the quantities figuring in equation (22); hence the definite sum will be

$$(24) \qquad \sum_{x=a}^{b} U_m^2 = C^2 h^{4m} \binom{2m}{m} \binom{N+m}{2m+1}.$$

It is useful to remark that this expression is independent of the origin of the variable x. Let us note also that $\sum U_0^2 = C^2 N$.

To determine the coefficient a_m from equation (15) it is still necessary to determine $\sum y \cdot U_m(x)$. For this purpose, let us start from equation (21′), that is from the *Newton*-series of U_m, with respect to the generalized binomial coefficient $\binom{x-a}{s}_h$. We have

$$(25) \quad \sum_{x=a}^{b} U_m y = C h^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{-N+m}{m-\nu} h^{-\nu} \sum_{x=a}^{b} \binom{x-a}{\nu}_h y.$$

In the analysis of continuous variables the quantity $\int_a^b x^s y \, dx$ is known as the moment of y of the s-th degree (we will say the s-th power-moment). We have seen that in analysing equidistant discontinuous variables it is not advantageous to operate with powers, but that it is far better to express the quantities in binomial coefficients. Indeed if an expression were given in power-series, we could not consider it to be a full solution, as it would still be advantageous to transform it into a binomial series. Therefore we have no use for power-moments and shall introduce binomial moments. The generalized *binomial moment* of degree ν will be denoted by $\mathcal{B}_\nu$ and defined by the following equation

$$(26) \qquad \sum_{x=a}^{b} \binom{x-a}{\nu}_h y = \mathcal{B}_\nu .$$

This can be easily expressed by ordinary binomial coefficients, for if we introduce a new variable $\xi=(x-a)/h$, we have

$$(27) \qquad \mathcal{B}_\nu = h^\nu \sum_{\xi=0}^{N} \binom{\xi}{\nu} y.$$

As will be shown later in (§ 9) there is a far better method for rapidly computing the binomial moments than there is in the case of power-moments.

Several statisticians have remarked that it is not advisable to introduce moments of high order into calculations. In fact, if N is large, these numbers will increase rapidly with the order of the moment, will become very large, and their coefficients in the formulae will necessarily become very small. It is very difficult to operate with such numbers, the causes of errors being many.

To obviate this inconvenience, I have introduced the *mean binomial moment;* that of the ν-*th* degree will be denoted by $\mathcal{T}_\nu$ and defined by

$$\mathcal{T}_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y \Big/ \sum_{x=a}^{b} \binom{x-a}{\nu}_h .$$

The sum figuring in the denominator is according to § 2 equal to

$$\frac{1}{h} \binom{b-a}{\nu+1}_h = \binom{N}{\nu+1} h^\nu ,$$

so that T_ν will be

$$(28) \qquad T_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y \Big/ \binom{N}{\nu+1} h^\nu .$$

If we introduce the variable ξ mentioned above, we obtain

$$T_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \Big/ \binom{N}{\nu+1} .$$

Consequently the mean binomial moments are independent of the origin of the variable x and of the interval h. The binomial moment increases rapidly with N and also with ν, though less rapidly than the ordinary power-moments. However, the mean binomial moment remains of the order of magnitude of y, whatever N or ν may be. For instance if $y=k$, it follows that $T_\nu=k$ for any value of ν.

Substituting in (25) the value of T_ν obtained in (28), we have

$$\sum_{x=a}^{b} U_m(x) y = C h^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m}\binom{-N+m}{m-\nu}\binom{N}{\nu+1} T_\nu .$$

To this expression we can give the following form,

$$(-1)^m C h^{2m} (m+1)\binom{N}{m+1} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+\nu}{m}\binom{m}{\nu} \frac{T_\nu}{\nu+1} .$$

To simplify we shall write

$$(29) \qquad \beta_{m\nu} = (-1)^{m+\nu} \binom{m+\nu}{m}\binom{m}{\nu} \frac{1}{\nu+1} .$$

These numbers are integers, and as they are very useful are presented in the following table, which presents all the numbers necessary for parabolas up to the tenth degree.

Table for $\beta_{m\nu}$

$m \backslash \nu$	0	1	2	3	4	5	6	7	8	9	10
1	-1	1									
2	1	-3	2								
3	-1	6	-10	5							
4	1	-10	30	-35	14						
5	-1	15	-70	140	-126	42					
6	1	-21	140	-420	630	-462	132				
7	-1	28	-252	1050	-2310	2772	-1716	426			
8	1	-36	420	-2310	6930	-12012	12012	-6435	1430		
9	-1	45	-660	4620	-18018	42042	-60060	51480	-24310	4862	
10	1	-55	990	-8580	42042	-126126	240240	-291720	218780	-92378	16796

The following relation can be used for checking these numbers:

$$\beta_{m0} + \beta_{m1} + \beta_{m2} + \cdots + \beta_{mm} = 0 .$$

Moreover, let us put

$$(30) \qquad \sum_{\nu=0}^{m+1} \beta_{m\nu} T_\nu = \Theta_m$$ [12]

If we already know the mean binomial moments, the values of Θ_m may readily be computed with the aid of the table above.

Finally we obtain

$$(31) \qquad \sum_{x=a}^{b} U_m(x) y = C h^{2m} (m+1)! \binom{N}{m+1} \Theta_m .$$

As this expression could be termed the orthogonal moment of degree m of y, we could consider Θ_m as a certain *mean ortho-*

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[12] $\beta_{m\nu}$ and Θ_m of this paper correspond respectively to $(-1)^m \beta_{m\nu}$ and $(-1)^m \Theta_m$ of the paper *loc. cit.* [9].

gonal moment of y of degree m. These quantities, as we shall see, are very important; they are independent of the origin, of the interval, and of the constant C. Thus, they are valid for all orthogonal polynomials.

In particular $\Theta_0 = T_0 = B_0/N$ is the arithmetic mean of the given quantities y.

Finally, equations (15), (24) and (31) yield, after simplification, the formula for a_m, that is,

$$a_m = \frac{(2m+1)\Theta_m}{Ch^{2m}\binom{N+m}{m}}. \tag{32}$$

The coefficient a_m is independent of the origin of x. In particular if $m=0$, we have $a_0 = \Theta_0/C$.

Now the equation of the approximating parabola is known, in the form of its expansion into a series of U_m polynomials (13), and our problem is solved. But if it is necessary to compute a table of the values of the parabola $f_n(x)$ corresponding to $x=a$, $a+h$, $a+2h$, ..., the corresponding values of $U_m(x)$ must first be calculated by formula (21′). Although this seems easy enough, yet if N is large, the computation is a tedious one even with the aid of Table IV which presents the values of the binomial coefficients, $\binom{\xi}{s}$ for integral values of ξ. If it should be necessary to compute $\binom{\xi}{s}$ for non-integer values of ξ, the calculation must be made in the ordinary way, that is, by multiplication At all events the calculation would not be shorter if the $U_m(x)$ were expanded into power-series. The labor will be decreased, however, if tables giving the values of $U_m(x)$ corresponding to $x=a$, $a+h$, $a+2h$, ..., are available. I adopted this procedure in my paper published in 1921 and later *Essher*, and also *Lorentz*, did the same[18].

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[18] See *loc. cit* [5], [9], [10], [7] and [8].

It will be shown, however, that these tables are superfluous, as by a *transformation* of formula (14) into a *Newton*-series, we can get the required values still more rapidly by the method of *addition of differences;* and if an interpolation is necessary for any values whatsoever of x , *Newton's* formula will give it in the shortest way.

Moreover, by this method we shall be independent of the value of the constant C , that is of the orthogonal polynomial chosen.

We will give in § 10 formulae leading directly to *Newton's* expansion, so that the computer will have nothing to do with the orthogonal polynomials themselves. He will only have to compute the binomial moments B_ν , and then deduce the mean orthogonal moments Θ_m, which will give, with appropriate coefficients, the solution in *Newton's* series.

§ 8. *Determination of the measure of obtained approximation.* The approximation is generally measured by the *mean-square-deviation* σ_n^2 , that is by the mean of the squares of deviations between the parabola and the given values y . It is expressed by

$$\sigma_n^2 = \frac{1}{N}\sum_{x=a}^{b}\left[f_n(x) - y\right]^2 .$$

If in this formula we put in place of $f_n(x)$ its expansion (14) in orthogonal polynomials, we shall obtain in consequence of the condition, (13) of orthogonality,

$$\sigma_n^2 = \frac{1}{N}\sum_{x=a}^{b} y^2 + \frac{1}{N}\sum_{m=0}^{n+1}\left[a_m^2 \sum_{x=a}^{b} U_m^2 - 2a_m \sum_{x=a}^{b} U_m\, y\right] ,$$

and on account of (13)

$$\sigma_n^2 = \frac{1}{N}\sum_{x=a}^{b} y^2 - \frac{1}{N}\sum_{m=0}^{n+1}\left[a_m^2 \sum_{x=a}^{b} U_m^2\right] . \tag{32'}$$

We can still simplify this result, since from formulae (24) and (32) it follows that

$$(33) \qquad \frac{1}{N} a_m^2 \sum_{x=a}^{b} U_m^2 = \frac{(2m+1)\binom{N-1}{m}\theta_m^2}{\binom{N+m}{m}}$$

And if to abbreviate we write

$$(34) \qquad C_m = (2m+1)\binom{N-1}{m} \Big/ \binom{N+m}{m}$$

and note that $c_0 = 1$, then the mean-square-deviation will be

$$(35) \quad \sigma_n^2 = \frac{1}{N}\sum_{x=a}^{b} y^2 - \theta_0^2 - C_1\theta_1^2 - C_2\theta_2^2 - \cdots - C_n\theta_n^2 .$$

The coefficients C_m can be easily calculated by (34), using Table IV for the binomial coefficients. But we shall see that C_m is equal to the absolute value of a certain quantity, which we have denoted by C_{mo} and which is given in Table III for values of N up to 100.

As the mean orthogonal moments are already known, therefore to obtain σ_n^2 it is sufficient to compute Σy^2.

Remark. All quantities figuring in (35) are independent of the origin, of the interval and of the constant C, consequently this formula is valid for all systems of orthogonal polynomials.

Sometimes it is necessary to know σ_{ns}^2, the mean-square of deviations between two parabolas, one of degree n, the other of degree s, (where $n > s$), both approximating to the same values of y; that is

$$\sigma_{ns}^2 = \frac{1}{N}\sum_{x=a}^{b} \left[f_n(x) - f_s(x) \right]^2 .$$

If in this formula we place the values of $f_n(x)$ and $f_s(x)$ expressed in series (13) of orthogonal polynomials, we have

$$f_n(x) - f_s(x) = a_{s+1} U_{s+1} + a_{s+2} U_{s+2} + \cdots + a_n U_n$$

and in consequence of (14)

$$\sigma_{sn}{}^2 = \frac{1}{N} \sum_{m=s+1}^{n+1} a_m^2 \sum_{x=a}^{b} U_m^2$$

then, using formulae (33) and (34) we find that

$$\sigma_{ns}^2 = C_{s+1}\,\Theta_{s+1}^2 + \cdots + C_n \Theta_n^2 .$$

Having obtained the equation of an approximating parabola of degree n, it may develop that the resultant approximation is not close enough, the mean-square-deviation being too large. We may then pass on to a parabola of degree $n+1$ by determining only the one additional coefficient a_{n+1}, —the others do not change. For this purpose we must compute T_{n+1} and then Θ_{n+1}; the coefficient will be given by (32). The new mean-square-deviation will be

$$\sigma_{n+1}{}^2 = \sigma_n^2 - C_{n+1} \Theta_{n+1}{}^2 .$$

Remark. If the binomial moments $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ were given, and we proceeded to determine a polynomial of degree n such that its first $n+1$ binomial moments should equal respectively to the values given above, thus employing the *principle of moments,* we should reach the same result as though we had imposed the principle of *least squares*. In the case of polynomials both principles lead to the same function.

§ 9. *Determination of the binomial moments. Chetverikoff* has given a very good method for their determination, which dispenses with all multiplication. We have seen that

$$B_\nu = \sum_{x=a}^{b} \binom{x-a}{\nu}_h y = h^\nu \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \, .$$

Chetverikoff's method produces the last sum, that is the ordinary binomial moment $(h=1)$; to obtain the generalized binomial moment B_ν this must still be multiplied by h^ν; but it is needless to carry out this multiplication, since we need only the mean-binomial-moment T_ν, which is given in both cases by

$$T_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} y \Big/ \binom{N}{\nu+1} .$$

The method consists in the following: Let us denote by $y(\xi)$ the value of y corresponding to ξ ; in the first column of a table the values of $y(\xi)$ are written in the reverse order of magnitude of ξ, that is

$$y(N-1),\ y(N-2),\ \cdots\cdot y(1),\ y(0) .$$

Into the first line of all the columns we write the same number, $y(N-1)$. Into the ν-th line of column μ we put the sum of the two numbers figuring in line $\nu-1$ of column μ, and in line ν of column $\mu-1$.

In column μ we stop at the line $N-\mu+2$; the number figuring there will be $\Sigma\binom{\xi}{\mu-2}y$; to obtain the mean binomial-moment of degree $\mu-2$ this must be divided by $\binom{N}{\mu-1}$. If we want the mean-binomial-moments $T_0, T_1, T_2, \cdots, T_n$, we must compute $n+1$ columns. An example is given in § 13.

An elementary demonstration of this method is given in the papers loc. cit. [9] and [10]. We will give here a more direct one. Let us denote by $\Phi(\nu,\mu)$ the number written into the ν-th line of column μ. The rule of computation will be

$$\Phi(\nu,\mu-1)+\Phi(\nu-1,\mu)=\Phi(\nu,\mu).$$

This is an equation of *partial differences* of the first order which may be written as follows:

$$(\alpha) \qquad \Phi(\nu+1,\mu+1)-\Phi(\nu+1,\mu)-\Phi(\nu,\mu+1)=0.$$

Let us solve it utilizing *Laplace's* method of *generating functions.*

We will call $u=u(t,\mu)$ the generating function of $\Phi(\nu,\mu)$ with respect to ν, if in the expansion of u in powers of t the coefficient of t^ν is equal to $\Phi(\nu,\mu)$, where μ is a parameter of u. Since $\Phi(0,\mu)=0$, then we have,

$$u=\Phi(1,\mu)t+\Phi(2,\mu)t^2+\cdots+$$
$$\Phi(\nu,\mu)t^\nu+\Phi(\nu+1,\mu)t^{\nu+1}+\cdots.$$

From this we easily deduce the generating function of $\Phi(\nu+1,\mu)$. If we divide both members by t, the coefficient of t^ν in the second member will be $\Phi(\nu+1,\mu)$. Hence $u(t,\mu)/t$ is the generating function sought. Since μ is a parameter, it follows from the preceding that the generating function of $\Phi(\nu,\mu+1)$ will be $u(t,\mu+1)$, and that of $\Phi(\nu+1,\mu+1)$ will be $u(t,\mu+1)/t$. If in equation (α) above we substitute the corresponding generating functions for the function Φ, we obtain

$$(\beta) \qquad (1-t)u(t,\mu+1)-u(t,\mu)=0.$$

In this way we have reduced the partial difference-equation (α) of the function Φ to an ordinary difference-equation of its generating function u. The solution expanded into powers of t, will give the function Φ itself, which is sought.

Equation (β) is a homogeneous linear equation with constant coefficients, t being only a parameter from this point of view, and can be solved immediately, yielding

$$(\gamma) \qquad u=\omega(t)/(1-t)^{\mu}$$

where $\omega(t)$ is an arbitrary function of t; and may be determined by the initial values, that is, by the values put into the first column. Placing $\mu=1$ into equation (γ) we obtain the generating function of these values. Hence,

$$\omega(t)/(1-t)=y(N-1)t+y(N-2)t^2+\cdots+y(0)t^N.$$

Finally we have

$$u=\Big[y(N-1)t+\cdots+y(N-z)t^z+\cdots \\ +y(0)t^N\Big](1-t)^{-\mu+1}.$$

Since

$$(1-t)^{-\mu+1}=\sum_{s=0}^{\infty}\binom{\mu-2+s}{\mu-2}t^s,$$

we may obtain the coefficient of t^{ν} in the expansion of u.

By letting $s=\nu-z$ we have therefore that

$$\Phi(\nu,\mu)=\sum_{z=1}^{N+1}\binom{\mu-2+\nu-z}{\mu-2}y(N-z).$$

If we place in this expression $\nu=N-\mu+2$ and $N-z=\xi$, we see that

$$\Phi(N-\mu+2,\mu)=\sum_{\xi=0}^{N}\binom{\xi}{\mu-2}y(\xi).$$

We conclude, therefore, that the number figuring in line $N-\mu+2$ of column μ will be the ordinary binomial moment of degree $\mu-2$. It was this that was to be demonstrated.

§ 10. *Transformation of the orthogonal series into Newton's expansion.* Since the approximating parabola and the mean square deviation are independent of the constant of the orthogonal polynomial used, it is natural to transform equation (13) so that it shall also be independent of this constant. This can be done by a transformation into *Newton's* series.

We have seen that the coefficients of *Newton's* series (4) are

$$f_n(a), \Delta f_n(a), \Delta^2 f_n(a), \cdots, \Delta^n f_n(a).$$

To obtain this series, therefore, it is sufficient to determine these quantities, starting from the orthogonal expansion.

Deriving from formula (13) the s-*th* difference of $f_n(x)$, we have

$$(36) \qquad \Delta^s f_n(x)=\sum_{m=s}^{n+1} a_m\,\Delta^s U_m(x).$$

To obtain $\Delta^s U_m$, we start from formula (21′). Since $\binom{x-a}{\nu-s}_h$ is a multiplying factor of the s-th difference of U_m, we conclude that for $x=a$, all terms obtained from (21′) will vanish except the term in which $\nu=s$. Hence we have for $s=0,1,2,\cdots,m$

$$\Delta^s U_m(a) = C h^{2m}\binom{m+s}{m}.$$

Placing this value, and that of a_m taken from (32), into equation (36) we get

$$\Delta^s f_n(a) = \sum_{m=s}^{n+1} (-1)^{m-s}(2m+1)\binom{m+s}{m}\frac{\binom{N-s-1}{m-s}\theta_m}{\binom{N+m}{m}},$$

where θ_m is the mean orthogonal moment of § 7. To abbreviate we shall write

$$C_{ms} = (-1)^{m-s}(2m+1)\binom{m+s}{m}\frac{\binom{N-s-1}{m-s}}{\binom{N+m}{m}}. \tag{37}$$

Finally we find that

$$\Delta^s f_n(a) = C_{ss}\theta_s + C_{s+1,s}\theta_{s+1} + \cdots + C_{ns}\theta_n. \tag{38}$$

For instance, we have for $s=0$,

$$f_n(a) = C_{00}\theta_0 + C_{10}\theta_1 + C_{20}\theta_2 + \cdots + C_{n0}\theta_n$$

where $C_{00}=1$. Let us remark again, that the second member of equation (38) is independent of the origin, of the interval, and of the constant C of the orthogonal polynomial chosen.

The importance of the numbers C_{ms} was first recognized by *W. Kviatovsszky,* who calculated a table for these numbers. This table has not been published. The author's Table III is more extensive, giving these numbers with more decimals for N up to 100, and for parabolas up to the seventh degree.

Having obtained, by the above method, the *Newton* expansion of the approximating parabola, it may happen that the expansion corresponding to a parabola of degree $n+1$ is desired, and this requires the calculation of Θ_{n+1}. The coefficients of the new binomial expansion of $f_{n+1}(x)$ are easily deduced from those of $f_n(x)$ previously obtained, since

$$\Delta^s f_{n+1}(a) = \Delta^s f_n(a) + C_{n+1,s}\,\Theta_{n+1}.$$

The work previously done is therefore not lost.

§ 11. *Method of the addition of differences.* Knowing the coefficients of *Newton's* formula $f_n(a), \Delta f_n(a), \cdots \Delta^n f_n(a)$, we can proceed to calculate a table of the values of $f_n(x)$ corresponding to $x=a,\ a+h,\ a+2h, \cdot\cdot, b-h$ by adding the differences. This method has been used by *H. Henning* in his remarkable paper on the Trend-lines.[14] It proceeds as follows. The function $f_n(x)$ being of degree n, it is evident that $\Delta^n f_n(x) = \Delta^n f_n(a)$ is a constant. Into the first line of the first column of a table we shall write the number $\Delta^{n-1} f_n(a)$. Into the other lines of the first column we put the number of the preceding line of the same column, increased by $\Delta^n f_n(a)$. We stop in this column at line $N-n+1$. According to *Newton's* formula, we have

$$\Delta^{n-1} f_n(x+h) = \Delta^{n-1} f_n(x) + \Delta^n f_n(x)$$

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[14] *H. Henning,* Die Analyse von Wirschaftskurven, Vierteljahreshefte zur Konjunkturforschung. Berlin 1927.

or

$$\Delta^{n-1} f_n(x) = \Delta^{n-1} f_n(a) + \xi \Delta^n f_n(a)$$

where $\xi = (x-a)/h$; hence the first column will contain the values of the $(n-1)$-th differences of $f_n(x)$. It is advisable, before continuing, to check the last number in this column by the formula above, putting therein $\xi = N-n$.

Into the first line of the μ-th column we write the number $\Delta^{n-\mu} f_n(a)$, and into the ν-th line of the same column the sum of the numbers figuring in line $\nu-1$ of column $\mu-1$ and column μ. The computation will be stopped in this column at the line $N-n+\mu$. The μ-th column will contain the values of the $(n-\mu)$-th differences of $f_n(x)$, which follows from *Newton's* formula

$$\Delta^{n-\mu} f_n(x+h) = \Delta^{n-\mu} f_n(x) + \Delta^{n-\mu+1} f_n(x)$$

or

$$\Delta^{n-\mu} f_n(x) = \Delta^{n-\mu} f_n(a) + \binom{\xi}{1} \Delta^{n-\mu+1} f_n(a) + \dots + \binom{\xi}{\mu} \Delta^n f_n(a). \tag{39}$$

Before going on, the last number of the column μ should be checked by the preceding formula, putting into it $\xi = N-n+\mu-1$.

We continue in the same manner, — the last column to be computed is the n-th and will contain the values of $f_n(a), f_n(a+h), \dots, f_n(b-h)$. This last number can be checked by formula (39), by making the substitution $\mu = n$ and $\xi = N-1$. An example is given in § 13.

§ 12. *Recapitulation of the operations.* To solve the problem of approximation of § 5, it is necessary first to compute the mean-binomial-moments $T_0, T_1, \dots, T_n$. This is done by drawing up

Chetverikoff's table (§ 9) and dividing the number in the line $N-\mu+2$ of column μ by $\binom{N}{\mu-1}$. We obtain in this way $T_{\mu-2}$: this must be repeated in every column.

Then the numbers $\beta_{m\nu}$ are taken from Table I., and the mean-orthogonal-moments $\theta_0,\ \theta_1, \cdots \theta_n$ are calculated by

$$(30)\quad \theta_0 = T_0,\quad \theta_m = \beta_{m0} T_0 + \beta_{m1} T_1 + \dots + \beta_{mm} T_m .$$

Then $\Sigma y^2/N$ is computed. The numbers $C_m = |C_{m0}|$ are taken from Table III, or if this table fails, calculated by formula (34) and by Table IV which gives the binomial coefficients. The mean-square-deviation is calculated by formula

$$(35)\quad \sigma_n^2 = \Sigma y^2/N - \theta_0^2 - C_1\theta_1^2 + \dots - C_n \theta_n^2 .$$

If this quantity is conveniently small, the approximation is considered close enough; if not, we proceed to calculate θ_{n+1}, and σ_{n+1}^2 and so on until a sufficiently small mean-square-deviation is reached.

Now we proceed to deduce the *Newton's* expansion of the required function $f_n(x)$. The numbers C_{ms} are taken from Table III, or if this table fails, calculated by formula (37) with the aid of Table IV.

The constants of *Newton's* formula are given by (38)

$$f_n(a) = \theta_0 + C_{10}\,\theta_1 + C_{20}\,\theta_2 + \dots + C_{n0}\,\theta_n$$

$$\Delta f_n(a) = C_{11}\theta_1 + C_{21}\theta_2 + \dots + C_{n1}\,\theta_n$$

$$\Delta^2 f_n(a) = C_{22}\,\Theta_2 + C_{32}\,\Theta_3 + \cdots + C_{n2}\,\Theta_n$$

$$- - - - - - - - - - - - - -$$

$$\Delta^s f_n(a) = C_{ss}\,\Theta_s + C_{s+1,\,s}\,\Theta_{s+1} + \cdots + C_{ns}\,\Theta_n$$

$$\Delta^n f_n(a) = C_{nn}\,\Theta_n$$

Now the equation of the parabola is known in the form of its *Newton*-series

$$f_n(x) = f_n(a) + \binom{\xi}{1}\Delta f_n(a) + \binom{\xi}{2}\Delta^2 f_n(a) + \cdots + \binom{\xi}{n}\Delta^n f_n(a) \tag{4}$$

where $\xi = (x-a)/h$. This will be considered as the desired solution. As has been said, it is quite useless to expand the parabola in powers of x.

If it is necessary to deduce a parabola of degree $n+1$ starting from equation (4); the new coefficients will be

$$f_{n+1}(a) = f_n(a) + C_{n+1,0}\,\Theta_{n+1}$$

$$\Delta^s f_{n+1}(a) = \Delta^s f_n(a) + C_{n+1,s}\,\Theta_{n+1}.$$

Generally a table of the values corresponding to the parabola and to $x = a, a+h, \cdots, b-h$, is needed; this will be computed by the method of addition of differences (§ 11). The last column will contain the required values.

§ 13. *Example 1.* Let us choose an example given by *Lorentz*[15] in which six values of y are given, and where $N=6$,

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[15] loc. cit. [8], p. 21.

$h=2$ and $a=-5$. The approximating parabolas of degrees 1, 2, 3, 4, and 5 are required.

The values of y corresponding to $x=-5,-3,-1,1,3,5$ are written in the first column of the table below in reversed order of magnitude of x. The other numbers of the table are obtained by *Chetverikoff's* method (§ 9).

12293	12293	12293	12293	12293	12293	12293
10875	23168	35461	47754	60047	72340	
10058	33226	68687	116441	176488		
10018	43244	111931	228372			
8530	51774	163705				
7880	59654					

The required *mean-binomial-moments* will be

$$T_0=59654/6=9942{,}3333^{*} \qquad T_3=176488/15=11765{,}8667$$

$$T_1=163705/15=10913{,}6667 \qquad T_4=72340/6=12056{,}6667$$

$$T_2=228372/20=11418{,}6 \qquad T_5=12293$$

The *mean-orthogonal-moments* are

$$\Theta_0=T_0=9942{,}3333$$

$$\Theta_1=T_1-T_0=971{,}3333$$

$$\Theta_2=2T_2-3T_1+T_0=38{,}3333$$

$$\Theta_3=5T_3-10T_2+6T_1-T_0=183$$

$$\Theta_4=14T_4-35T_3+30T_2-10T_1+T_0=351{,}6667$$

$$\Theta_5=42T_5-126T_4+140T_3-70T_2+15T_1-T_0=-1152$$

The squares of the mean-orthogonal-moments are

$$\Theta_0^2=98849985{,}48 \qquad \Theta_3^2=33489$$

$$\Theta_1^2=943488{,}38 \qquad \Theta_4^2=123669{,}45$$

$$\Theta_2^2=1484{,}82 \qquad \Theta_5^2=1327104$$

$$\Sigma y^2/6=100960410{,}3$$

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*Editor's note. In this country we usually write $\frac{69654}{6}=9942.3333$ instead of $9942{,}3333$. For the purposes of this paper I prefer to use the continental notation appearing in Professor Jordan's manuscript. I agree that the following typical product of three factors $\frac{1}{k}\cdot 341{,}77395\cdot\binom{x+5}{1}$ appearing on page 300 is less liable to be confused than if written $\frac{1}{k}\cdot 341.77395\cdot\binom{x+5}{1}$.

From Table III we get:

$C_{10} = -2,14285714$ $\qquad$ $C_{40} = 0,21428571$

$C_{20} = 1,78571429$ $\qquad$ $C_{50} = -0,02380952$

$C_{30} = -0,83333333$

Now the mean-square-deviations for the parabolas of degree 0, 1, 2, 3, 4, 5 will be:

$$
\begin{aligned}
\sigma_0^2 &= \Sigma y^2/6 - \theta_0^2 = 2110424,82 \\
\sigma_1^2 &= \sigma_0^2 - C_1\theta_1^2 = 88664,05 \\
\sigma_2^2 &= \sigma_1^2 - C_2\theta_2^2 = 86012,58 \\
\sigma_3^2 &= \sigma_2^2 - C_3\theta_3^2 = 58105,08 \\
\sigma_4^2 &= \sigma_3^2 - C_4\theta_4^2 = 31604,49 \\
\sigma_5^2 &= \sigma_4^2 - C_5\theta_5^2 = 6,80
\end{aligned}
$$

As the parabola of degree 5 passes through the given six points, therefore σ_5^2 should be exactly equal to zero.

Let us now proceed to the determination of *Newton's* formula corresponding to the five parabolas required. It is to be observed that the amount of work required to solve this problem, is independent of the number of observations. First the following numbers must be taken from Table III:

$C_{11} = 0,85714286$ $\qquad$ $C_{22} = 1,07142857$ $\qquad$ $C_{43} = -3$

$C_{21} = -2,14285714$ $\qquad$ $C_{32} = -2,5$ $\qquad$ $C_{53} = 1,33333333$

$C_{31} = 2$ $\qquad$ $C_{42} = 1,92857143$ $\qquad$ $C_{44} = 3$

$C_{41} = -0,85714286$ $\qquad$ $C_{52} = -0,5$ $\qquad$ $C_{54} = -3$

$C_{51} = 0,14285714$ $\qquad$ $C_{33} = 1,66666667$ $\qquad$ $C_{55} = 6$

Formulae (38) of the preceding paragraph give:

$$f_1(a) = \Theta_0 + C_{10}\,\Theta_1 \qquad = 7860,90480$$

$$f_2(a) = f_1(a) + C_{20}\,\Theta_2 \qquad = 7929,71427$$

$$f_3(a) = f_2(a) + C_{30}\,\Theta_3 \qquad = 7777,21427$$

$$f_4(a) = f_3(a) + C_{40}\,\Theta_4 \qquad = 7852,57142$$

$$f_5(a) = f_4(a) + C_{50}\,\Theta_5 \qquad = 7779,99998$$

$$\Delta f_1(a) = C_{11}\,\Theta_1 \qquad = 832,57140$$

$$\Delta f_2(a) = \Delta f_1(a) + C_{21}\,\Theta_2 \qquad = 750,00005$$

$$\Delta f_3(a) = \Delta f_2(a) + C_{31}\,\Theta_3 \qquad = 1116,00005$$

$$\Delta f_4(a) = \Delta f_3(a) + C_{41}\,\Theta_4 \qquad = 814,57114$$

$$\Delta f_5(a) = \Delta f_4(a) + C_{51}\,\Theta_5 \qquad = 650,00002$$

$$\Delta^2 f_2(a) = C_{22}\,\Theta_2 \qquad = 41,28568$$

$$\Delta^2 f_3(a) = \Delta^2 f_2(a) + C_{32}\,\Theta_3 \qquad = -416,21432$$

$$\Delta^2 f_4(a) = \Delta^2 f_3(a) + C_{42}\,\Theta_4 \qquad = 262,00003$$

$$\Delta^2 f_5(a) = \Delta^2 f_4(a) + C_{52}\,\Theta_5 \qquad = 838,00003$$

$$\Delta^3 f_3(a) = C_{33}\,\Theta_3 \qquad = 305,00000$$

$$\Delta^3 f_4(a) = \Delta^3 f_3(a) + C_{43}\,O_4 \qquad = -705,00010$$

$$\Delta^3 f_5(a) = \Delta^3 f_4(a) + C_{53}\,\Theta_5 \qquad = -2286,00010$$

$$\Delta^4 f_4(a) = C_{44}\,\Theta_4 \qquad = 1055,00010$$

$$\Delta^4 f_5(a) = \Delta^4 f_4(a) + C_{54}\,\Theta_5 = \qquad 4511,00010$$

$$\Delta^5 f_5(a) = C_{55}\,\Theta_5 \qquad = -6912$$

Before writing the formulae of the parabolas, let us introduce $(x-a)/h=\xi$ and put $f(a+\xi h)=F(\xi)$, we have then

$$F_1(\xi)=7860{,}905+832{,}571\binom{\xi}{1}$$

$$F_2(\xi)=7929{,}714+750\binom{\xi}{1}+41{,}286\binom{\xi}{2}$$

$$F_3(\xi)=7777{,}214+1116\binom{\xi}{1}-416{,}214\binom{\xi}{2}+305\binom{\xi}{3}$$

$$F_4(\xi)=7852{,}571+814{,}571\binom{\xi}{1}+262{,}1\binom{\xi}{2}-705\binom{\xi}{3}+1055\binom{\xi}{4}$$

$$F_5(\xi)=7780+650\binom{\xi}{1}+838\binom{\xi}{2}-2286\binom{\xi}{3}+4511\binom{\xi}{4}-6912\binom{\xi}{5}$$

These results are exact to three decimal places.

The method of the addition of differences can be applied immediately to these equation.

Example 2. The values corresponding to the approximating parabola of the fifth degree and corresponding to the given values of x in the preceding example are to be determined

Starting from the last equation above, we will determine $F_5(\xi)$ for $\xi=0,1,2,3,4,5$ Noting that $\Delta^5F(0)=-6912$ the table below is obtained by using the method described in § 11 The last column contains the required values of $F_5(\xi)$, which, as in this case the parabola passes through the given points, should be exactly equal to the given values y in Example 1

4511	−2286	838	650	7880
−2401	2225	−1448	1488	8530
	−176	777	40	10018
		601	817	10050
			1418	10875
				12293

The results are exact to three decimal places.

Remark on the number of decimals to which the calculations are to be carried out If for instance a parabola of the fifth degree is to be determined approximating the values of y, which are given with a precision of half a unit, then $\Delta^5f(a)$ should be calculated to seven decimals if the number of the given values is near 50, or to eight decimals if it is near one hundred; $\Delta^4 f(a)$ should be calculated to six or seven respectively; $\Delta^3 f(a)$ to five or six, and so on. The corresponding orthogonal moments and the numbers C_{ms} must be of course calculated to the same number of decimals.

Example 3. Changing the origin. The given values y in Example 1 correspond to

$$\text{July } 1, 1901 + \xi$$

for $\xi = 0, 1, 2, \cdot\cdot 5$, where ξ is expressed in years. These values have been approximated by $F_5(\xi)$, obtained in the last equation of Example (1), to abbreviate we shall write it $F(\xi)$. In this case, July 1, 1901 corresponds to $a = 0$. It is required to develop the function $F(\xi)$ in a series of binomial coefficients $\binom{x-c}{\nu}$, where $c = -\frac{1}{2}$ corresponds to January 1, 1901. Equation (7) will give for $h = 1$ and $c - a = -\frac{1}{2}$ with an exactitude of three decimals

$$\Delta F(-\tfrac{1}{2}) = -6912$$

$$\Delta^4 F(-\tfrac{1}{2}) = 4511 + \tfrac{1}{2} \cdot 6912 = 7967$$

$$\Delta^3 F(-\tfrac{1}{2}) = -2286 - \tfrac{1}{2} \cdot 4511 - \tfrac{3}{8} \cdot 6912 = -7133{,}5$$

$$\Delta^2 F(-\tfrac{1}{2}) = 838 + \tfrac{1}{2} \cdot 2286 + \tfrac{3}{8} \cdot 4511 + \tfrac{5}{16} \cdot 6912 = 5832{,}625$$

$$\Delta F(-\tfrac{1}{2}) = 650 - \tfrac{1}{2} \cdot 838 - \tfrac{3}{8} \cdot 2286 - \tfrac{5}{16} \cdot 4511$$
$$- \tfrac{35}{128} \cdot 6912 = -3925{,}938$$

$$F(-\tfrac{1}{2}) = 7780 - \tfrac{1}{2} \cdot 650 + \tfrac{3}{8} \cdot 838 + \tfrac{5}{16} \cdot 2286$$
$$+ \tfrac{35}{128} \cdot 4511 + \tfrac{63}{256} \cdot 6912 = 11418{,}102$$

Finally the required formula will be

$$F(\xi)=11418,102-3925,938\binom{\xi+\frac{1}{2}}{1}+5832,625\binom{\xi+\frac{1}{2}}{2}$$

$$-7133,5\binom{\xi+\frac{1}{2}}{3}+7967\binom{\xi+\frac{1}{2}}{4}-6912\binom{\xi+\frac{1}{2}}{5}.$$

This equation was checked by calculating $F(0)$ which was necessarily equal to 7780.

Example 4. Changing the interval. Let us suppose the following function given

$$f(x)=7780+\frac{650}{2}\binom{x+5}{1}_2+\frac{838}{2^2}\binom{x+5}{2}_2-\frac{2286}{2^3}\binom{x+5}{3}_2$$

$$+\frac{4511}{2^4}\binom{x+5}{4}_2-\frac{6912}{2^5}\binom{x+5}{5}_2.$$

This is a *Newton*-expansion in which the decrement of the generalized binomial coefficient and the increment of the differences are both $h=2$. It is required to deduce another *Newton*-expansion such that the mentioned decrement and increment should both be $k=1/3$.

For this purpose it is necessary to calculate $\Delta^s f(x)$ for $x=-5$ and, $s=1,2,3,4,5$. First we must determine the numbers A_m^s/h^m introduced in § 4.

For $h=2$, and $k=1/3$ we have

$A_1^1/h=1/6$ $\qquad$ $A_2^1/h^2=-5/72$

$A_3^1/h^3=55/1296$ $\qquad$ $A_4^1/h^4=-935/31104$

$A_5^1/h^5=21505/933120$

$A_2^2/h^2 = 1/36$ $\qquad A_3^2/h^3 = -5/216$

$A_4^2/h^4 = 295/15552$ $\qquad A_5^2/h^5 = -55/3456$

$A_3^3/h^3 = 1/216$ $\qquad A_4^3/h^4 = -5/864$

$A_5^3/h^5 = 185/31104$ $\qquad A_4^4/h^4 = 1/1296$

$A_5^4/h^5 = -5/3888$ $\qquad A_5^5/h^5 = 1/7776$

Now we can proceed to calculate the differences $\Delta^s f(a)$ given by the formula in § 4. We have

$$\Delta f(a) = \frac{1}{6}\cdot 650 - \frac{5}{72}\cdot 838 - \frac{55}{1296}\cdot 2286 - \frac{935}{31104}\cdot 4511 - \frac{21505}{933120}\cdot 6912 = -341{,}77395$$

$$\Delta^2 f(a) = \frac{1}{36}\cdot 838 + \frac{5}{216}\cdot 2286 + \frac{295}{15552}\cdot 4511 + \frac{55}{3456}\cdot 6912 = 271{,}76190$$

$$\Delta^3 f(a) = \frac{-1}{216}\cdot 2286 - \frac{5}{864}\cdot 4511 - \frac{185}{31104}\cdot 6912 = -77{,}79977$$

$$\Delta^4 f(a) = \frac{1}{1296}\cdot 4511 + \frac{5}{3888}\cdot 6912 = 12{,}36960$$

$$\Delta^5 f(a) = -\frac{6912}{7776} = -0{,}88889$$

Hence

$$f(x) = 7780 - \frac{1}{k}\cdot 341{,}77395\binom{x+5}{1} + \frac{1}{k^2}\cdot 271{,}76190\binom{x+5}{2}$$

$$-\frac{1}{k^3}(77,79977)\binom{x+5}{3}+\frac{1}{k^4}(12,3696)\binom{x+5}{4}$$

$$-\frac{1}{k^5}\cdot(0,88889)\binom{x+5}{5}$$

where $k=1/3$. If we want to apply the method of the addition of differences it is preferable to change the variable by introducing $\xi=(x-a)/k$ and writing $F(\xi)=f(a+\xi k)$. We have

$$F(\xi)=7780-341,77395\binom{\xi}{1}+271,7619\binom{\xi}{2}$$

$$-77,79977\binom{\xi}{3}+12,3696\binom{\xi}{4}-0,88889\binom{\xi}{5}.$$

Of course it would have been better to change the variable before beginning to calculate the numbers A_m^s . but we wanted to show the method in its generality. It is advisable to check the above equation by putting into it $\xi=6$: the result must be $F(6)=f(1)=8430$. The checking has given this number with a precision of 5 decimal places.

Starting from $\Delta^5F(\xi)=-0,88889$, let us determine the first values of $F(\xi)$ for $\xi=0,1,2,3,\cdots6$, by the method of the addition of differences

12,36960	− 77,79977	271,76190	− 341,77395	7780
11,48071	− 65,43017	193,96213	− 70,01205	7438,22605
10,59182	− 53,94946	128,53196	123,95008	7368,21400
	− 43,35764	74,58250	252,48204	7492,16408
		31,22486	327,06454	7744,64612
			358,28940	8071,71066
				8430,00006

§ 14. *First problem of correlation.* It is required to determine the coefficient of correlation r_{nm} between the deviations $f_n(x)-y$ of a parabola of degree m approximating to the values y (§ 3) and of the deviations $f'_m(x) - y'$ of a parabola of degree m approximating to other values y' corresponding to the same x quantities. Let us suppose that $n \geqq m$.

The definition of this coefficient is the following:

$$(40) \quad r_{nm} = \frac{1}{N \sigma_n \sigma'_m} \sum_{x=a}^{b} \left[f_n(x) - y \right] \left[f'_m(x) - y' \right]$$

where σ_n^2 is the mean-square-deviation of $f_n(x)$ and y (§ 8); and σ'^2_m the mean-square-deviation of $f_n(x)$ and y'.

Let us put into (40) the values of $f_n(x)$ and $f'_m(x)$ expanded into series of orthogonal polynomials (14); then the sum in the second member of (40) will be

$$(41) \quad \Sigma y y' - \Sigma y (a'_0 U_0 + a'_1 U_1 + \cdots + a'_m U_m)$$

$$- \Sigma y' (a_0 U_0 + a_1 U_1 + \cdot \cdot + a_n U_n)$$

$$+ a_0 a'_0 \Sigma U_0^2 + a_1 a'_1 \Sigma U_1^2 + \cdot \cdot + a_m a'_m \Sigma U_m^2 .$$

This expression may be simplified; indeed, starting from equation (15) we have, after multiplication by a'_s

$$a'_s \Sigma y U_s = a_s a'_s \Sigma U_s^2 .$$

Putting into this equation successively $s = 0, 1, 2, \cdots m$, and adding the results, we obtain the values of the second sum of (41). To have the third, we start from the equation analogous to (15)

$$\Sigma y' U_s = a'_s \Sigma U_s^2$$

and multiply both members by a_s; putting successively $s = 0, 1, 2, \cdots, n$ and adding the results, we obtain an expression for the third sum of (41). Finally this formula will be

$$\Sigma yy' - a_0 a_0' \Sigma U_0^2 - a_1 a_1' \Sigma U_1^2 - \cdots - a_n a_n' \Sigma U_n^2.$$

We have still to determine the quantity $a_s a_s' \Sigma U_s^2 / N$; but according to (24) and (32) this expression is equal to $C_m \Theta_m \Theta_m'$, where Θ_m' is the mean-orthogonal-moment of degree m of y'. We conclude that,

$$(42) \quad r_{nm} = \frac{1}{\sigma_n \sigma_m'} \left[\frac{1}{N} \Sigma yy' - \Theta_0 \Theta_0' - C_1 \Theta_1 \Theta_1' - \cdots - C_n \Theta_n \Theta_n' \right].$$

Sometimes, n being given, the coefficients of correlation r_{nm} are sought for all values of $m = 0, 1, 2, \cdots, n$. In this case we will first divide the quantity within the brackets in equation (42) by σ_n and then divide the quotient successively by the values of σ_m' for $m = 0, 1, 2, \cdots, n$

All the quantities figuring in equation (42) are known from the previous determination of the approximating functions $f_n(x)$ and $f_m'(x)$, except $\Sigma yy'/N$. To obtain the desired coefficient of correlation, it is necessary to compute this additional last quantity.

Formula (42) also shows the importance of the mean-orthogonal moments of the given quantities y and y'; it is independent of the origin, of the interval, and of the constant of the orthogonal polynomial chosen.

The most important particular case of r_{nm} is r_{00}; this being the coefficient of correlation of the deviations of y and of y' from their respective averages. Thus r_{00} shows the simultaneity of the variation from the respective averages. Another important particular case is r_{11}, which gives the correlation between $f_1(x)$-y

and $f'_1(x)-y'_1$ — thus measuring the simultaneity of periodical deviations from the respective linear trend-lines.

If n and m are large, the approximating parabolas will follow the principal secular and periodical variations, will therefore have nearly the same maxima and minima as the values y and y'. In this event the remaining deviations will be mainly due to chance, and thus the coefficient of correlation loses its importance.

§ 15. *Second problem of correlation.* Given the functions $f_n(x)$ and $f_\nu(x)$, of degree n and ν respectively, approximating the values of a quantity y. Let us denote by ξ the deviations of the two corresponding parabolas:

$$\xi = f_n(x) - f_\nu(x).$$

Moreover, let us likewise have given the functions $f'_m(x)$ and $f'_\mu(x)$ of degree m and μ respectively approximating the values of another quantity y', and denote by η the deviations between the corresponding two parabolas.

It is required to find the coefficient of correlation $r_{n\nu,\,m\mu}$ between the deviations ξ and η. According to the definition of th coefficient of correlation we have

$$(43) \quad r_{n\nu,\,m\mu} = \frac{1}{N\sigma_{n\nu}\sigma_{m\mu}} \sum_{x=a}^{b} \left[f_n(x) - f_\nu(x)\right]\left[f'_m(x) - f'_\mu(x)\right]$$

where $\sigma_{n\nu}^2$ denotes the mean-square-deviation of ξ, and $\sigma'^{2}_{m\mu}$ the mean-square-deviation of η (§ 8). Both are known from the determination of the approximating parabolas.

Let us suppose that $n \geqq m > \nu \geqq \mu$. Substituting, for $f_n(x)$, $f_\nu(x)$, $f'_m(x)$ and $f'_\mu(x)$ their expansions in orthogonal polynomials (14), the sum in the second member of (43) will be

$$\Sigma(a_{\nu+1}U_{\nu+1}+\cdots+a_nU_n)(a'_{\mu+1}U_{\mu+1}+\quad+a'_mU_m).$$

This yields, in consequence of the orthogonality of the polynomial U_s,

$$a_{\nu+1}a'_{\nu+1}\Sigma U_{\nu+1}^2+\cdot\quad+a_ma'_m\Sigma U_m^2.$$

As we have seen in the preceding paragraph,

$$\frac{1}{N}a_sa'_s\Sigma U_s^2=C_s\Theta_s\Theta'_s.$$

Hence

$$(44)\quad r_{n\nu,m\mu}=\frac{1}{\sigma_{n\nu}\sigma_{m\mu}}\left[C_{\nu+1}\Theta_{\nu+1}\Theta'_{\nu+1}+\quad\cdot+C_m\Theta_m\Theta'_m\right].$$

All the quantities of the second member of this equation are known from the previous parabolic approximation, so that the calculating of the coefficient (44) is easy. We note that it is a simple function of the mean-orthogonal moments, of the mean-square deviations and of the number C_s. (formula 34 or table III).

If m and μ are given and this coefficient must be computed for several values of n and ν, we first divide the quantity within the brackets of formula (44) by $\sigma_{m\mu}$ and then divide the quotient successively by the different values of $\sigma_{n\nu}$.

In the second problem also, the most important particular case is that of $r_{n0,m0}$, especially if n and m are large, in which case the deviations $f_n(x)-y$ and $f'_m(x)-y'$ could be considered as negligible. In this case the coefficient of correlation (44) of the trend-lines is much more important than the coefficient of correlation (42) of the trend-deviations.

Example on correlation. *A. Sipos* has determined trend-lines up to the third degree of Hungarian impotts and exports in 1882-1913.[16] The mean orthogonal moments for imports were

$$\Theta_0 = 1254,25938 \qquad \Theta_2 = 70,63941$$
$$\Theta_1 = 206,02671 \qquad \Theta_3 = 21,27341$$

The mean-square-deviations corresponding to the parabolas of imports of degree 0, 1, 2, 3 were

$$\sigma_0 = 383,777 \qquad \sigma_2 = 83,552$$
$$\sigma_1 = 166,317 \qquad \sigma_3 = 69,330$$

The equation of the third degree parabola of approximation was

$$f_3(x) = 864,12484 + 20,38562\binom{x}{1} - 2,82064\binom{x}{2} + 0,45504\binom{x}{3}$$

where $x = 0$ corresponds to 1882 and $f_3(x)$ is given in million gold crowns.

The corresponding values for exports were

$$\Theta'_0 = 1234,4 \qquad \Theta'_2 = 37,80645$$
$$\Theta'_1 = 192,675 \qquad \Theta'_3 = 5,58515$$

$$\sigma'_0 = 37,587 \qquad \sigma'_2 = 58,480$$
$$\sigma'_1 = 96,662 \qquad \sigma'_3 = 57,184$$

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[16] *A. Sipos*, Praktische Anwendung der Trendberechnungsmethode von Jordan. Mitteilungen der Ungarischen Landeskommission für Wirtschaftsstatistik und Konjunkturforschung. Budapest 1930.

The equation of the third degree parabola of exports was

$$f_3'(x) = 821,24103 + 15,09955\binom{x}{1} + 0,28944\binom{x}{2} + 0,11947\binom{x}{3}.$$

First problem of correlation. Let us determine the more important particular cases of the coefficient of correlation r_{nm}, between the deviations $y - f_n(x)$ of import and the deviations $y' - f_m'(x)$ of export.

The coefficient of correlation between the deviations of the given quantities and their respective averages is

$$r_{00} = \frac{1}{\sigma_0 \sigma_0'}\left[\frac{1}{N}\Sigma y y' - \theta_0 \theta_0'\right],$$

and since $\frac{1}{N}\Sigma y y' = 1672457,80$ we have

$$r_{00} = 0,9586$$

This correlation is a very strong one.

The coefficient of correlation between the deviations of the given quantities and their respective linear trend-lines is

$$r_{11} = \frac{1}{\sigma_1 \sigma_1'}\left[\frac{1}{N}\Sigma y y' - \theta_0 \theta_0' - C_1 \theta_1 \theta_1'\right] = 0,7669.$$

(The number C_1 was taken from table III. $C_1 = 2,81818\cdots$) This correlation is still strong enough.

The coefficient of correlation between the deviations of the given quantities and their respective third degree trend-lines is

$$r_{33} = \frac{1}{\sigma_3 \sigma_3'}\left[\frac{1}{N}\Sigma y y' - \theta_0 \theta_0' - C_1 \theta_1 \theta_1' - C_2 \theta_2 \theta_2' - C_3 \theta_3 \theta_3'\right]$$
$$= 0,1739.$$

This correlation is already very small, the deviations are mainly due to chance. From Table III we had $C_2 = 4{,}14438503$ and $C_3 = 4{,}80748663$.

Second problem of correlation. The coefficient of correlation $r_{n\nu, m\mu}$ between the deviations of two trend-lines of import of degree n and ν respectively and the deviations of two trend-lines of export of degree m and μ respectively, is to be determined. We will only consider the most important particular case, the correlation between the third degree trend-lines and the respective averages, that is

$$r_{30,30} = \frac{1}{\sigma_{30}\sigma'_{30}} \left[C_1 \theta_1 \theta'_1 + C_2 \theta_2 \theta'_2 + C_3 \theta_3 \theta'_3 \right]$$

where

$$\sigma_{30}^2 = C_1 \theta_1^2 + C_2 \theta_2^2 + C_3 \theta_3^2 = 142659{,}2203$$

and

$$\sigma_{30}'^2 = C_1 \theta_1'^2 + C_2 \theta_2'^2 + C_3 \theta_3'^2 = 110695{,}9458.$$

And therefore $\sigma_{30} = 377{,}70256$ and $\sigma'_{30} = 332{,}71000.$

The quantities in the brackets have already been calculated in the first problem, so that we have

$$r_{30,30} = 0{,}9828.$$

The obtained value is very near to that of r_{00} calculated above, proving that the third degree trend-lines well represent the given quantities. This would not be true in the case of the second degree trend-lines; in fact we should obtain $r_{20,20} = 0,9865$ widely different from r_{20} obtained before.

§ 16. *Some mathematical properties of the orthogonal polynomials. Symmetry of the polynomial* U_m. If we substitute in formula (18) $a+b-h-x$ for x, we get

$$U_m(a+b-h-x) = Ch^m \Sigma \binom{m}{s}\binom{b-h+sh-x}{s}\binom{a-h-x}{m-s}$$

$$= Ch^m \Sigma \binom{m}{s}\binom{x-b}{s}\binom{x-a+mh-sh}{m-s}$$

and putting $s = m-\mu$, it follows that

$$U_m(a+b-h-x) = (-1)^m U_m(x). \tag{45}$$

This formula shows the symmetry of the polynomial.

Let us consider the particular case,

$$x-a = \frac{1}{2}(b-a-h) = kh.$$

We have, then,

$$b-a = (2k+1)h, \qquad N = 2k+1$$

$$x-a = kh \qquad x-b = -(k+1)h.$$

From (45) it follows that

$$U_m(a+kh) = (-1)^m U_m(a+kh).$$

Hence we have

$$U_{2m+1}(a+kh)=0.$$

From equation (45) we easily obtain

$$\Delta U_m(x)=(-1)^{m+1}\Delta U_m(a+b-2h-x)$$

and

$$\Delta^s U_m(x)=(-1)^{m+s}\Delta^s U_m(a+b-sh-x).$$

Function-equation of $U_m(x)$. This can be deduced in the following way: let us develop xU_m into a series of orthogonal polynomials; we find that

$$(46) \qquad xU_m(x)=A_{m-1}U_{m-1}+A_mU_m+A_{m+1}U_{m+1}$$

as, in consequence of the orthogonality of these polynomials, the other terms vanish; indeed, if μ is different from $m-1$, m and $m+1$, it follows that

$$\sum_{x=a}^{b} xU_mU_\mu=0.$$

Since equation (46) holds for every value of x, and $U_m(x)$ is known for three particular values of x, we can determine the coefficients A_s.

We know these values, since equation (21) gives for $x=b$

$$U_m(b)=C(m)h^m\binom{\frac{b-a}{h}+m}{m}$$ [17]

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[17] As C can be dependent of m, we will write $C(m)$ in the following formulae, instead of C.

and for $x=a$, after changing a into b and inversely

$$U_m(a)=C(m)(-1)^m h^m \binom{\frac{b-a-h}{h}}{m}.$$

Moreover, in consequence of the above-mentioned symmetry of the polynomials, we have

$$U_m(b-h)=(-1)^m U_m(a).$$

The two last equations give by using formula (46)

$$(47) \quad aU_m(a)=A_{m-1}U_{m-1}(a)+A_m U_m(a)+A_{m+1}U_{m+1}(a)$$

and

$$(48) \quad \begin{aligned}(b-h)U_m(b-h)=A_{m-1}U_{m-1}(b-h)+A_m U_m(b-h)\\ +A_{m+1}U_{m+1}(b-h).\end{aligned}$$

Multiplying both sides of equation (47) by $(-1)^m$ and adding it to (48), we find that

$$(b+a-h)U_m(b-h)=2A_m U_m(b-h)$$

and

$$A_m=\tfrac{1}{2}(b+a-h).$$

Multiplying both sides of (47) by $(-1)^m$ and subtracting it from (48) yields

$$(b-a-h)U_m(b-h)=2A_{m-1}U_{m-1}(b-h)+2A_{m+1}U_{m+1}(b-h).$$

Since

$$U_m(b-h)=C(m)h^m\binom{b-a-h}{m}$$

we find that

$$\begin{aligned}\tfrac{1}{2}(b-a-h)C(m)h\,\frac{b-a-mh}{m}&=A_{m-1}C(m-1)\\ &\quad+A_{m+1}C(m+1)h^2\,\frac{(b-a-mh)(b-a-mh-h)}{m(m+1)}\,.\end{aligned} \tag{49}$$

As, in consequence of the symmetry of the polynomials, we have

$$U_m(b)=(-1)^mU_m(a-h)=C(m)h^m\binom{b-a+mh}{m}$$

hence we can deduce two other equations analogous to (47) and (48), *i.e.*,

$$b\bar{U}_m(b)=A_{m-1}U_{m-1}(b)+A_mU_m(b)+A_{m+1}U_{m+1}(b)$$

and

$$(a-h)U_m(a-h)=A_{m-1}U_{m-1}(a-h)+A_mU_m(a-h)+A_{m+1}U_{m+1}(a-h).$$

After multiplying the second equation by $(-1)^m$ and subtracting the result from the first, we obtain

$$\tfrac{1}{2}(b-a+h)U_m(b)=A_{m-1}U_{m-1}(b)+A_{m+1}U_{m+1}(b)$$

or

$$(50)\qquad \tfrac{1}{2}(b-a+h)C(m)h\,\frac{b-a+mh}{m}=A_{m-1}C(m-1)$$
$$+A_{m+1}C(m+1)h^2\,\frac{(b-a+mh)(b-a+mh+h)}{m(m+1)}\,.$$

Finally subtracting (50) from (49), and simplifying, we have

$$A_{m+1}=\frac{(m+1)^2}{2(2m+1)}\;\frac{C(m)}{hC(m+1)}\,.$$

We deduce A_{m-1} from (49) by substituting therein the above value for A_{m+1}, yielding

$$A_{m-1}=\frac{(b-a)^2-m^2h^2}{2(2m+1)}\;\frac{hC(m)}{C(m-1)}\,.$$

Now the function-equation (46) is known.

We might have proceeded in another way. Having obtained A_m by using the equations (47) and (48), we could determine, for instance A_{m+1} in the usual way by multiplying both members of equation (46) by U_{m+1}, and summing x from a to b,

$$\sum_{x=a}^{b} U_{m+1} U_m x = A_{m+1} \sum_{x=a}^{b} U_{m+1}^2 .$$

Since ΣU_{m+1}^2 is already known from (24), we need only determine the first member and this may be done by applying formula (3) to the quantity

$$\Sigma U_m (x U_m).$$

Difference-equation of $U_m(x)$. We will start from

$$\Delta U_{m+1} = C \Delta^{m+2} \left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h \right] .$$

According to equation (1) we have

$$\Delta \left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h \right] = h \binom{x-a}{m}_h \binom{x-b}{m+1}_h + h \binom{x-b}{m}_h \binom{x-a+h}{m+1}_h .$$

Therefore

$$\Delta U_{m+1} = \frac{hC}{m+1} \Delta^{m+1} \left\{ (2x-a-b-mh+h) \cdot \left[\binom{x-a}{m+1}_h \binom{x-b}{m+1}_h \right] \right\} .$$

Applying to this expression formula (2), giving the $m+1$th difference of a product, it follows that

$$(51)\quad \Delta U_{m+1} = \frac{h}{m+1}\Big[(2x-a-b+mh+3h)\Delta U_m + 2h(m+1)U_m\Big].$$

Now let us deduce a second formula for ΔU_{m+1}. We can write this quantity in the following manner

$$\Delta U_{m+1} = C\Delta^{m+2}\left[\binom{x-a}{m+1}_h\binom{x-b}{m+1}_h\right] =$$

$$\frac{C}{(m+1)^2}\,\Delta^{m+2}\Big\{\Big[x(x-h)+x(h-a-b-2mh)$$

$$+(a+mh)(b+mh)\Big]\cdot\left[\binom{x-a}{m}_h\binom{x-b}{m}_h\right]\Big\}.$$

Again using formula (2) to deduce the $(m+2)$-th difference of the preceding product, we have after simplification

$$\Delta U_{m+1} = \frac{1}{(m+1)^2}\Big[(x-a+2h)(x-b+2h)\Delta^2 U_m$$

$$+(m+2)h(2x+3h+a-b)\Delta U_m + (m+1)(m+2)h^2 U_m\Big].$$

Finally, taking into account (51), this equation results in

$$(x-a+2h)(x-b+2h)\Delta^2 U_m + \left[2x-a-b+3h-m(m+1)h\right]h\Delta U_m$$

(52)

$$-m(m+1)h^2 U_m = 0.$$

This is the required difference-equation; it is a linear equation of the second order and can be solved by *Boole's* method.[18] If we put $\xi = (x-a)/h$ the solution will be

$$(53)\quad U_m = Ch^{2m}\sum_{\nu=0}^{m+1}(-1)^{\nu}\binom{m+\nu}{\nu}\binom{m+N}{m-\nu}\binom{\xi+\nu}{\nu}.$$

This expression of U_m differs from those we obtained in paragraph 6.

The roots of $U_m(x)=0$. *L. Fejér*[19] has demonstrated the following theorems concerning these roots:

The roots of $U_m(x)=0$ are all real and single, they are all situated in the interval

$$a,\ b-h$$

Whatever ξ may be, in the interval $a+\xi h,\ a+\xi h+h$ there is at most one root of

$$U_m(x)=0.$$

Fejér showed moreover that if $g_m(x)$ is a polynomial of degree m, and if in its *Newton* expansion the coefficient of $\binom{x}{m}$

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[18] *Boole*, Calculus of Finite Differences, 1860, p. 176.

[19] See the Appendix in loc. cit. [5]

is equal to unity, the polynomial which minimizes the following expression

$$\sum_{x=a}^{b} \left[g_m(x) \right]^2$$

is the orthogonal polynomial $U_m(x)$, with the constant C suitably chosen.* Indeed the first conditions of a minimum are that

$$\sum_{x=a}^{b} \binom{x}{\nu} g_m(x) = 0 \quad \text{for} \quad 0 \leqq \nu < m$$

and these are identical with the conditions of orthogonality (14) The second condition of the minimum is always satisfied in these cases, as has been shown in § 5.

§ 17. *Graduation by orthogonal polynomials.* Let us consider an odd number of consecutive values of x, say $2k+1$, and the corresponding values of y. A smoothed value of y is wanted for the central term, viz. for $x = a+kh$. This will be obtained by determining, according to the principle of least squares, a parabola of degree m, so that the sum of the squares of deviations between the parabola and the points x, y shall be a minimum.

The equation of the parabola expressed in orthogonal polynomials (13) will be

$$f_n(x) = a_0 + a_1 U_1(x) + \cdots + a_n U_n(x)$$

and the smoothed value required is given by $f_n(a+kh)$.

In consequence of the symmetry of the polynomials, formula

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*See Essher's first polynomial in § 21.

(45), we have $U_{2m+1}(a+kh)=0$, so that the equation of the parabola will give

$$f_n(a+kh)=a_0+a_2U_2(a+kh)+a_4U_4(a+kh)+\dots$$

From this formula we see that it is useless to consider parabolas of odd degree, as for instance a parabola of the second degree will give the same smoothed value for y as would a parabola of the third degree. Therefore we will consider only parabolas of even degree.

The values of $U_{2m}(a+kh)$ are given by our formulae (18), (21), etc., but we can obtain a much simpler formula for them, starting from the *Function-Equation* of $U_{2m}(x)$ given by formula (46), which, since $A_m=a+kh$, we can write in the following manner

$$\frac{(2k+1)^2-m^2}{2(2m+1)}\,\frac{U_{m-1}}{C(m-1)}+(a+kh-x)\,\frac{U(m)}{h^3C(m)}$$

$$+\frac{(m+1)^2}{2(2m+1)}\,\frac{U_{m+1}}{h^4C(m+1)}=0\,.$$

This holds for every value of x. For $x=a+kh$ the term in U_{2m} will vanish and we have

$$U_{m+1}(a+kh)=-\frac{2k+1-m}{m+1}\,\frac{2k+1+m}{m+1}\,\frac{C(m+1)}{C(m-1)}\,h^4U_{m-1}(a+kh).$$

This equation can be solved by putting into it successively $m=1,2,3,\dots$ It follows that

$$U_2(a+kh) = -k(k+1)C(2)h^4$$

$$U_4(a+kh) = \binom{k}{2}\binom{k+2}{2}C(4)h^8$$

and so on

$$U_{2m}(a+kh) = (-1)^m h^{4m} C(2m)\binom{k}{m}\binom{k+m}{m}.$$

For a_{2m} we have found in § 7

$$a_{2m} = \frac{(4m+1)\Theta_{2m}}{C(2m)h^{4m}\binom{2k+1+2m}{2m}},$$

Hence if to abbreviate we write

$$(54) \qquad S_{2m} = (-1)^m(4m+1)\binom{2m}{m}\frac{\binom{k+m}{2m}}{\binom{2k+1+2m}{2m}}$$

we have

$$a_{2m}U_{2m} = S_{2m}\,\Theta_{2m}$$

and finally the required smoothed value of y is given by

$$(55) \qquad f_{2m}(a+kh) = \Theta_0 + S_2\Theta_2 + S_4\Theta_4 + \cdots + S_{2m}\Theta_{2m}.$$

This formula is also independent of the interval and of the constant of the orthogonal polynomial used.

The values of S_{2m} necessary up to parabolas of the tenth degree and up to 29 ordinates are given in Table II, so the calculation of the graduated value is very simple. All we need do is to compute the mean binomial moments by *Chetverikoff's* method (§ 9) and calculate the corresponding mean orthogonal moments Θ_{2m} by formula (30). This must of course be repeated for every value which is to be graduated.

Example 7. Nine point graduation, employing second and fourth degree parabolas. The given values are

x	y	x	y
0	2502	5	2904
1	2548	6	3064
2	2597	7	3188
3	2675	8	3309
4	2770		

The mean binomial moments were calculated by the method of § 9, and were found to be

$$T_0 = 2839{,}66667 \qquad T_3 = 3182{,}21429$$

$$T_1 = 3014{,}97222 \qquad T_4 = 3225{,}87302$$

$$T_2 = 3117{,}16667$$

Hence the mean orthogonal moments are

$$\Theta_0 = T_0 = 2839{,}66667 \qquad \Theta_4 = 14T_4 - 35T_3 + 30T_2 - 10T_1$$

$$\Theta_2 = 2T_2 - 3T_1 + T_0 = 29{,}08335 \qquad + T_0 = -10{,}33148$$

From Table II. we take $S_2 = -1{,}81818182$ and $S_4 = 1{,}13286713$.

TABLE II. GRADUATION.

N	S_2	S_4	S_6	S_8	S_{10}
3	− 1				
5	− 1,4285 7143	0,4285 7142 9			
7	− 1,6666 6667	0,8181 8181 8	− 0,1515 1515 2		
9	− 1,8181 8182	1,1328 6713	− 0,3636 3636 4	0,0489 5104 90	
11	− 1,9230 7694	1,3846 1538	− 0,5882 3529 4	0,1417 0040 5	− 0,0150 0364 37
13	− 2	1,5882 3529	− 0,8049 5356 0	0,2631 5789 5	− 0,0508 8167 99
15	− 2,0588 2353	1,7554 1796	− 1,0061 9195	0,4004 5766 6	− 0,1068 5152 8
17	− 2,1052 6316	1,8947 3684	− 1,1899 3135	0,5446 2242 6	− 0,1794 0503 4
19	− 2,1428 5714	2,0124 2236	− 1,3565 2174	0,6898 5507 2	− 0,2644 6776 6
21	− 2,1739 1304	2,1130 4348	− 1,5072 4638	0,8325 8370 8	− 0,3583 1116 7
23	− 2,2	2,2	− 1,6436 7816	0,9707 0819 4	− 0,4578 4204 7
25	− 2,2222 2222	2,2758 6207	− 1,7673 9587	1,1030 7749	− 0,5606 2291 4
27	− 2,2413 7931	2,3426 0289	− 1,8798 6652	1,2291 4349	− 0,6647 9271 2
29	− 2,2580 6451	2,4017 5953	− 1,9824 0469	1,3487 3583	− 0,7689 6251 1

Therefore the smoothed value with a parabola of the second degree will be

$$f_2(4) = \Theta_0 + S_2 \Theta_2 = 2786,78785$$

In the paper loc. cit.[9] (p. 31) these values have been approximated by a parabola of the third degree, the value obtained for $f_3(4)$ was (p. 33) $2786,78763$ according to what has been said, this value should be equal to the obtained results for $f_2(4)$.

Finally the smoothed value corresponding to a parabola of the fourth degree is

$$f_4(4) = f_2(4) + S_4 \Theta_4 = 2775,0837$$

BIBLIOGRAPHICAL AND HISTORICAL NOTES.[20]

§ 18. It was *Chebisheff* who first introduced orthogonal polynomials with respect to a discontinuous variable. He[21] especially treated the case of non-equidistant variables, from the mathematical point of view, in a very interesting manner, but his results were necessarily complicated. As we consider here only equidistant variables, this paper will not be discussed.

But *Chebisheff* also investigated the case of equidistant polynomials in two of his papers. In the first "Sur l'Interpolation par la Méthode des Moindres Carrés"[22] he denotes the orthogonal polynomial of degree m by $\varphi_m(x)$; the variable x taking values differing by unity, from $-\frac{1}{2}(N-1)$ to $\frac{1}{2}(N-1)$ inclusive. His polynomials can be obtained for instance from our formula (21′) by putting therein $h=1$, $a=-\frac{1}{2}(N-1)$ and $C(m!)^2$. Formula (24) will give $\Sigma\varphi_m^2$ by putting into it $h=1$ and $C=(m!)^2$, where $m!$ stands for $1,2,3,\cdots m$.

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[20] The numbers of the formulae quoted refer to the present paper.

[21] Sur les Fractions Continues. 1855. Oeuvres, T. I. p. 201.

[22] Oeuvres 1859. Oeuvres, Tome I. p. 474.

In the second paper "Interpolation des valeurs equidistantes" in which he introduces such polynomials,[23] he denotes the polynomial of degree m by $\phi_m(x)$ and the variable x takes the values 1, 2, 3, ... $(n-1)$. These polynomials can also be obtained from formula (21′) and $\Sigma\phi_m^2$ from (24), by putting in them $h=1$, $a=1$, $N=n-1$ and $C=(m!)^2$.

§ 19. *J. P. Gram* utilized orthogonal polynomials for graduation according to the principle of least squares.[24] He denotes the polynomial of degree m by $\psi_m(x)$, the variable x taking the values

$$-\tfrac{1}{2}(N-1), \ldots, -1, 0, 1, \cdots, \tfrac{1}{2}(N-1)$$

where N is an odd number. Then writing

$$\psi_m(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m$$

he determines the coefficients by formulae

$$\sum_{x=-\frac{1}{2}(N-1)}^{\frac{1}{2}(N+1)} x^s \psi_m(x) = 0 \quad \text{for} \quad 0 \leqq s < m.$$

There are m such equations and $m+1$ unknown coefficients α_s one of them therefore will be arbitrary. *Gram* disposes of the first coefficient which is different from zero in such a manner that all the values of $\psi_m(x)$ corresponding to the considered values of x shall be integers and as small as possible.

For instance in the case of $\psi_1(x)$, it follows that $\alpha_0 = 0$, and in order to have $\psi_1(x) = x$, he puts $\alpha_1 = 1$.

Practically he uses only polynomials of the first, second and third degrees; and gives tables for $\psi_2(x)$ and $\psi_3(x)$ available for $N = 7, 9, 11, \cdots, 21$ and containing the corresponding values of

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[23] 1875. Oeuvres, T. II, p. 270. 219.

[24] Ueber partielle Ausgleichung mittelst Orthogonalfunktionen, Bulletin de l'Association des Actuaires Suisses, 1915.

$$\Sigma\psi_1^2, \Sigma\psi_2^2, \Sigma\psi_3^2.$$

The values of $\psi_2(x)$ can be obtained, for instance, from our formula (21′), if we put therein $h=1$, $a=-\frac{1}{2}(N-1)$, and either $C=-1$, if $(N-1)(N-2)$ is not divisible by three, or $C=-\frac{1}{3}$ if it is so divisible.

The values of $\psi_3(x)$ are also obtained from the same formula by putting into it $h=1$, $a=-\frac{1}{2}(N-1)$ and either $C=-1/4$, if $(N-2)(N-3)$ is not divisible by five, or $C=-1/20$, if it is so divisible. $C=1/2$ corresponds to $\psi_1(x)=x$.

Formula (24) would give the values of $\Sigma\psi_1^2$, $\Sigma\psi_2^2$ and $\Sigma\psi_3^2$ of the table, if we were to substitute $h=1$, and for C the respective values above-mentioned.

In order to give an example, *Gram* calculates the smoothed value corresponding to an eleven-point parabola of the second degree (p. 12). His calculation is short enough, but our general method is still shorter, as may be judged from the following determination of the smoothed central value of his example.

In the first column we have written the values of y corresponding to a reversed order of magnitude of x; the other columns have been obtained by simple addition, as indicated in § 9.

194	194	194	194
179	373	567	761
212	585	1152	1913
124	709	1861	3774
780	1489	3350	7124
000	1489	4839	11963
504	1993	6832	18795
244	2237	9069	27864
000	2237	11306	39170
582	2819	14125	...
000	2819	...	...

Hence

$$T_0 = \Theta_0 = 2819/11 = 256,272727\ldots$$

$$T_1 = 14125/55 = 256,81818\ldots$$

$$T_2 = 39170/165 = 237,393939\ldots$$

and

$$\Theta_2 = 2T_2 - 3T_3 + T_0 = -39,3939..$$

To have the smoothed value, we take from Table II.,

$$S_2 = -1,92307694$$

and it follows that the required smoothed value is

$$f_2(a+kh) = \Theta_0 + S_2\Theta_2 = 332,0303$$

agreeing with *Gram's* result

Although the calculation has been executed to nine figures, it is a very short one.

§ 20. *Ch. Jordan* "Sur une série de polynomes dont chaque somme partielle représente la meilleure approximation d'un degré donné suivant la méthode des moindres carrés."[25] (1921) In this paper the author treats the mathematical theory of orthogonal polynomials for equidistant values in the general case. Many of their mathematical properties are demonstrated: formulae, difference-equations, function-equations of these polynomials are given and some interesting propositions concerning their roots are demonstrated. Two particular polynomials were introduced. In the first, denoted by $Q_m(x)$, x takes the values $a, a+h, a+2h, \cdots, b-h$, where $b=a+Nh$; and N is an integer. The constant C was chosen as

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[25] Proceedings of the London Mathematical Society, Vol. XX., p. 297.

$$C = m!/2^m h^m.$$

This was done, as has been mentioned, so that for $a=-1$, $b=1$ and $h=0$ the limit of the polynomial $Q_m(x)$ should be identical with *Legendre's* polynomial.

The second particular case denoted by $q_m(x)$ was obtained from $Q_m(x)$ by putting $h=1$, $a=0$ and $b=N$. There are tables in this paper giving the values of $q_m(x)$ for $m=1, 2, 3, 4, 5$, for $N=m+1, m+2, \cdots, 20$ and for the values of $x=0, 1, 2, \cdots, N-1$. Moreover there is a table giving Σq_m^2 for $m=1, 2, 3, 4, 5$ and for $N=m+1, m+2, \cdots, 20$.

The problem of approximation was solved in this paper by formula (14) of the present work in the following manner:—the coefficients a_m were calculated from formula (13) above: $\Sigma y \cdot q_m(x)$ was computed first by multiplying every value of y by the corresponding values of $q_m(x)$ taken from the tables mentioned and then the products were added. The quantity Σq_m^2 was taken likewise from the tables.

The coefficients a_m being known, the mean square deviation was calculated by formula (32′). And finally, the required values of $f_n(x)$ were obtained by using formula (14), and by taking the necessary values of $q_m(x)$ from the tables.

In a *second* paper "Berechnung der Trendlinie auf Grund der Theorie der kleinsten Quadrate"[20] the determination of the coefficients a_m has been much simplified. These were obtained by multiplying the binomial moments by certain numbers, and were easily calculated with the aid of a table of binomial coefficients $\binom{N}{\nu}$. These tables exhibit the values given for N up to 55 and for ν up to ten, and are sufficient for parabolas up to the tenth degree. Although the calculation of ΣU_m^2 was not necessary for the determination of a_m, nevertheless it had to be evaluated for the determination of the mean square deviation. For this purpose a very simple formula was given for ΣQ_m^2 (p. 45).

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[20] Mitteilungen der Ungarischen Landeskommission für Wirtschaftsstatistik und Konjunkturforschung. Budapest, 1930.

In this paper the method of approximation by orthogonal polynomials has been freed from the tables giving the values of these polynomials corresponding to the given x values. These tables would be very voluminous if we wanted them extended up to one hundred observations and to parabolas up to the tenth degree.

This has been attained by giving formulae which permit us to pass easily from the orthogonal expansion of the approximating parabola to its *Newton* series, which gives directly the required values by means of the method of addition of the differences and by calculating the coefficients a_m without the evaluation of $\Sigma y\, Q_m(x)$.

The third paper "Sur la détermination de la tendance séculaire des grandeurs statistiques par la méthode des moindres carrés."[27] (1930) published somewhat later than the second, differs from it inasmuch as it introduces the mean orthogonal moments, giving (p. 585) a table for their calculation for parabolas up to the tenth degree,—the present table I. The coefficients a_m, the quantities $\frac{a_m^2}{N}\Sigma Q_m^2$ figuring in the formula of the mean square deviation, and those of $\frac{a_m a_{m'}\Sigma Q_m^2}{N}$ figuring in that of the different coefficients of correlation, are expressed by these moments. The calculation of ΣQ_m^2 became needless, which is very fortunate since these numbers are very large if N is large, and it would therefore be difficult to operate with such numbers. The table of the binomial coefficients has been extended sufficiently for one hundred observations (N up to 105).

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[27] Journal de la Sociátè Hongroise de Statistique, 1929, T. VII. p. 567.

Finally, in the *present* paper the orthogonal polynomials are left in the general form, the arbitrary constant remains entirely in the background, the coefficients a_m are no longer calculated, the mean orthogonal moments alone are used, and by the aid of these quantities the Newton expansion of the approximating parabola and also the mean square deviation and the coefficients of correlation are directly obtained. All unnecessary matter has been cleared away.

Table II, giving numbers S_{2m} useful for graduation, and Table III, rendering it easier to establish *Newton's* formula, are new. Table IV, of the binomial coefficients has been extended by a few lines (up to 110) in order to suffice for parabolic approximation of the tenth degree by the new formulae.

§ 21. *M. F. Essher* "Ueber die Sterblichkeit in Schweden 1886-1914"[28] denotes in this paper the orthogonal polynomial of degree m by $P_m(x)$; x taking the values of $-\frac{1}{2}(N-1),\cdots,\frac{1}{2}(N-1)$ Since the coefficient of x^m is taken as equal to unity, therefore the constant C of the polynomial $U_m(x)$ is, according to formula (21'), $C=m!/\binom{2m}{m}$. Putting this value into (21'), and writing $h=1, a=-\frac{1}{2}(N-1)$, it will give the values of *Essher's* polynomial corresponding to a given x. Formula (24) gives for the above value of C and $h=1$

$$\sum_{x=a}^{b} P_m^2 = \frac{(m!)^2}{\binom{2m}{m}} \binom{N+m}{2m+1}$$

In a *second* paper "On Graduation according to the Method of Least Squares by Means of Certain Polynomials"[29], *Essher* has employed other orthogonal polynomials, denoting them by $X_m(x)$ the variable x taking the values of $1,2,3,\ldots,N$. Adopting *Lorenz's* point of view (§ 22), he chose the constant C in such a manner that ΣX_m^2 should be equal to N. From our formula (24) we conclude that in this case

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[28] Medelanden fran Lunds Astronomiska Observatorium, Lund 1920.

[29] Försäkringsaktiebolaget Skandia 1855-1930, Stockholm, 1930. p. 107.

$$C = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\;.$$

In this way the expression ΣX_m^2 becomes very simple, but the polynomials themselves become complicated.

Putting $h=1$, $a=1$, and the above value of C into formula (21′) we have

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\sum_{\nu=0}^{m+1}\binom{x-N-1}{\nu}\binom{m+\nu}{m}\binom{N+m}{m-\nu}.$$

The coefficient a_m in an expression by *Essher's* polynomials is expressed very simply by our method. Indeed from (32) we get, if we put in this equation $h=1$ and the value or C above,

$$a_m = \Theta_m\sqrt{C_m}\;,$$

where Θ_m is the mean orthogonal moment of degree m and C_m the number given by formula (34), or $C_m = |C_{m0}|$ taken from Table III.

As a comparison let us determine the graduated value corresponding to Age 52, *Essher's* example 10. (p. 116)

The first column below contains the given values corresponding to x in an inverted order of magnitude; the other columns are obtained by the method of § 9. The graduation for the central value will be obtained by an eleven-point parabola of the second degree.

674	674	674	674
873	1547	2221	2895
1005	2552	4773	7668
1216	3768	8541	16209
1331	5099	13640	29849
1239	6338	19978	49827
1640	7978	27956	77783
1385	9363	37319	115102
1366	10729	48048	163150
1315	12044	60092	
851	12895		

Let us remark, that these numbers figure in *Essher's* table (p. 117). We shall have

$$T_0 = \Theta_0 = 12895/11 = 1172,272727\ldots$$
$$T_1 = 60092/55 = 1092,581818\ldots$$
$$T_2 = 163150/165 = 988,7878\ldots$$

Hence

$$\Theta_2 = 2T_2 - 3T_1 + T_0 = -127,896969\ldots$$

From Table II. we take $S_2 = -1,92307694$ and finally the required graduated value will be

$$\Theta_0 + S_2\Theta_2 = 1418,22$$

agreeing with *Essher's* result.

§ 22. *P. Lorentz* in the first edition of his paper "Der Trend"[80] introduced orthogonal polynomials, distinguishing two cases according as the number of observations was either even or odd. He denoted polynomials of degree m by $X_m(x)$ and chose them so that ΣX_m^2 should be equal to N.

If m is odd the variable takes the values

$$-\tfrac{1}{2}(N-1), \ldots, -1, 0, 1, \ldots \tfrac{1}{2}(N-1)$$

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[80] Vierteljahreshefte zur Konjunkturforschung. Sonderheft 9, Berlin, 1928.

and the value of C in $U_m(x)$ corresponding to the above condition, taken from (24), is

$$C = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}$$

Hence X_m is given by formula (21′) by putting in it the above value of C and placing $h=1$, $a=-\frac{1}{2}(N-1)$, so that we have

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\sum_{\nu=0}^{m+1}\binom{m+\nu}{m}\binom{N+m}{m-\nu}\binom{x-N-1}{\nu},$$

If m is even, the variable in $X_m(x)$ takes the values

$$-(N-1), -(N-3), \cdots -1, 1, 3, \cdots, (N-1)$$

and the value of C corresponding to the condition $\Sigma X_m^2 = N$ will be obtained from (24) by putting $h=2$, so that

$$C = \frac{1}{2^{2m}}\sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}$$

The polynomial is given by (21′) by placing in it this value of C and $h=2$, $a=-(N-1)$. Hence it follows that

$$X_m(x) = \sqrt{\frac{N}{\binom{2m}{m}\binom{N+m}{2m+1}}}\sum_{\nu=0}^{m+1}\frac{1}{2^{\nu}}\binom{m+\nu}{m}\binom{N+m}{m-\nu}\binom{x-N-1}{\nu}_2$$

Whether N be odd or even, the coefficient of a_m is given by our formula (32), the same formula appearing in *Esscher's* expansion, *i.e.*,

$$a_m = \theta_m \sqrt{C_m} \, .$$

Here θ_m is the mean orthogonal moment of degree m, and C_m is given by formula (34), or since $C_m = |C_{m0}|$, by Table III.

The paper contains five decimal tables giving $X_m(x)$ corresponding to the necessary integer values of x up to $m = 5$ and N up to 60. There are also other tables useful for the transformation of the orthogonal series into a power series, and also a table enabling one to change the interval of one year into that of one month.

The second edition of the paper does not differ in principle from the first. The polynomials remain the same, but the tables for $X_m(x)$ have been extended for m up to six, and for N up to eighty.

The only advantage that *Lorentz's* method possesses over ours is that when applying *Chetverikoff's* method to the determination of binomial moments the calculation in his system is a little easier, since the numbers to be added contain one or two figures less. But as this operation is generally made by calculating machines, this is but a slight advantage, and this is largely compensated for in the subsequent operations.

As an example, let us determinate the coefficients corresponding to *Lorentz's* polynomials in the orthogonal expansion of the example in § 13. There, the mean orthogonal moments found were

$\theta_0 = 9942,3333$ $\qquad$ $\theta_3 = 183$

$\theta_1 = 971,3333$ $\qquad$ $\theta_4 = 351,666$

$\theta_2 = 38,5333$ $\qquad$ $\theta_5 = -1152$

We have seen that $C_0 = 1$; the other numbers $C_m = |C_{m0}|$ are taken from Table II., their square roots being

$\sqrt{C_1} = 1,463\ 850\ 109$ $\quad$ $\sqrt{C_2} = 1,336\ 306\ 210$

$\sqrt{C_3} = 0,912\ 870\ 929$ $\quad$ $\sqrt{C_4} = 0,462\ 910\ 050$

$\sqrt{C_5} = 0,154\ 303\ 350$

Finally, the required coefficients a_m are given by our formula

$$a_m = \Theta_m \sqrt{C_m}$$

$a_0 = 9942,33333$ $\quad$ $a_1 = 1421,88641$

$a_2 = 51,49233$ $\quad$ $a_4 = 162,79003$

$a_5 = -177,57459$

in accordance with *Lorentz's* results.

The corresponding mean square deviations would be given by

$$\sigma_m^2 = \frac{1}{N} \Sigma y^2 - a_0^2 - a_1^2 - \cdots - a_m^2 \,.$$

Charles Jordan

TABLE III

These numbers are given to ten figures, although generally fewer will be sufficient, especially if as is generally the case the mean orthogonal moments are all of the same order of magnitude. In this event, a fixed number of *decimals,* properly chosen will suffice.

The numbers C_{ms} given are checked by the relation:

$$\sum_{s=0}^{m+1} C_{ms} = 2m+1$$

Remark.

$$\lim_{N=\infty} C_{m0} = 2m+1 \quad \text{and} \quad \lim_{N=\infty} C_{ms} = 0 \quad \text{if} \quad s \neq 0.$$

N	C_{10}	C_{11}	C_{20}	C_{21}
3	-1,5	1,5	0,5	-1,5
4	-1,8	1,2	1	-2
5	-2	1	1,428 571 429	-2,142 857 143
6	-2,142 857 143	0,857 142 857 1	1,785 714 286	-2.142 857 143
7	-2,25	0,75	2,083 333 333	-2,083 333 333
8	-2,333 333 333	0,666 666 666 7	2,333 333 333	-2
9	-2,4	0,6	2,545 454 545	-1,909 090 909
10	-2,454 545 455	0.545 454 545 5	2,727 272 727	-1,818 181 818
11	-2.5	0.5	2,884 615 385	-1,730 769 231
12	-2,538 461 538	0,461 538 461 5	3,021 978 021	-1,648 351 648
13	-2,571 428 571	0,428 571 428 6	3,142 857 143	-1,571 428 571
14	-2,6	0,4	3,25	-1,5
15	-2,625	0,375	3,345 588 235	-1,433 823 529
16	-2,647 058 823	0,352 941 176 4	3,431 372 549	-1,372 549 020
17	-2,666 666 667	0,333 333 333 3	3,508 771 930	-1,315 789 474
18	-2,684 210 526	0,315 789 473 6	3,578 947 368	-1,263 157 894
19	-2,7	0,3	3,642 857 143	-1,214 285 714
20	-2,714 285 714	0,285 714 285 7	3,701 298 702	-1,168 831 169
21	-2,727 272 727	0,272 727 272 7	3,754 940 711	-1,126 482 213
22	-2,739 130 436	0,260 869 565 2	3,804 347 826	-1,086 956 522
23	-2,75	0,25	3,85	-1,05
24	-2,76	0,24	3,892 307 692	-1,015 384 615
25	-2,769 230 769	0,230 769 230 8	3,931 623 933	-0,982 905 982 9
26	-2,777 777 778	0,222 222 222 2	3,968 253 969	-0,952 380 952 4
27	-2,785 714 286	0,214 285 714 2	4,002 463 055	-0,923 645 320 5
28	-2,793 103 447	0,206 896 551 6	4,034 482 758	-0,896 551 723 9
29	-2,8	0,2	4,064 516 128	-0,870 967 741 8
30	-2,806 451 614	0,193 548 387 1	4,092 741 935	-0,846 774 193 4
31	-2,8125	0,1875	4,119 318 182	-0,823 863 636 4
32	-2,818 181 818	0,181 818 181 8	4,144 385 026	-0,802 139 037 4
33	-2,823 529 411	0,176 470 588 2	4,168 067 225	-0.781 512 605 1
34	-2,828 571 429	0,171 428 571 4	4,190 476 192	-0,761 904 761 9
35	-2,833 333 333	0,166 666 666 7	4,211 711 712	-0,743 243 243 3
36	-2,837 837 838	0,162 162 162 2	4,231 863 442	-0,725 462 304 4
37	-2,842 105 263	0,157 894 736 8	4,251 012 146	-0,708 502 024 3
38	-2,846 153 846	0,153 846 153 8	4,269 230 769	-0,692 307 692 3
39	-2,85	0,15	4,286 585 369	-0,676 829 268 3
40	-2,853 658 537	0,146 341 463 4	4,303 135 888	-0,662 020 905 9
41	-2,857 142 857	0,142 857 142 9	4,318 936 877	-0,647 840 531 5
42	-2,860 465 116	0,139 534 883 7	4,334 038 055	-0,634 249 471 4
43	-2,863 636 364	0,136 363 636 4	4,348 484 848	-0,621 212 121 2
44	-2,866 666 667	0,133 333 333 3	4,362 318 840	-0,608 695 652 1
45	-2,869 565 217	0,130 434 782 6	4,375 578 168	-0,596 669 750 2
46	-2,872 340 425	0,127 659 574 6	4,388 297 872	-0,585 106 382 9
47	-2,875	0,125	4,400 510 204	-0,573 979 591 8
48	-2,877 551 021	0,122 448 979 6	4,412 244 898	-0,563 265 306 1
49	-2,88	0,12	4,423 529 411	-0,552 941 176 4
50	-2,882 352 941	0,117 647 058 8	4,434 389 140	-0,542 986 425 4

N	C_{10}	C_{11}	C_{20}	C_{21}
51	-2,884 615 385	0,115 384 615 4	4,444 847 606	-0,533 381 712 7
52	-2,886 792 453	0,113 207 547 2	4,454 926 625	-0,524 109 014 7
53	-2,888 888 889	0,111 111 111 1	4,464 646 465	-0,515 151 515 2
54	-2,890 909 091	0,109 090 909 1	4,474 025 974	-0,506 493 506 5
55	-2,892 857 143	0,107 142 857 1	4,483 082 706	-0,498 120 300 7
56	-2,894 736 842	0,105 263 157 9	4,491 833 031	-0,490 018 148 9
57	-2,896 551 724	0,103 448 275 9	4,500 292 227	-0,482 174 167 2
58	-2,898 305 085	0,101 694 915 3	4,508 474 576	-0,474 576 271 2
59	-2,9	0,1	4,516 393 442	-0,467 213 114 7
60	-2,901 639 344	0,098 360 655 74	4,524 061 343	-0,460 074 034 9
61	-2,903 225 806	0,096 774 193 54	4,531 490 014	-0,453 149 001 5
62	-2,904 761 905	0,095 238 095 24	4,538 690 476	-0,446 428 571 4
63	-2,906 250	0,093 750	4,545 673 077	-0,439 903 846 2
64	-2,907 692 307	0,092 307 692 30	4,552 447 552	-0,433 566 433 6
65	-2,909 090 909	0,090 909 090 91	4,559 023 067	-0,427 408 412 5
66	-2,910 447 761	0,089 552 238 80	4,565 408 253	-0,421 422 300 3
67	-2,911 764 706	0,088 235 294 12	4,571 611 254	-0,415 601 023 1
68	-2,913 043 478	0,086 956 521 74	4,577 639 752	-0,409 937 888 2
69	-2,914 285 714	0,085 714 285 72	4,583 501 007	-0,404 426 559 4
70	-2,915 492 960	0,084 507 042 26	4,589 201 879	-0,399 061 032 9
71	-2,916 666 667	0,083 333 333 33	4,594 748 858	-0,393 835 616 4
72	-2,917 808 219	0,082 191 780 82	4,600 148 094	-0,388 744 909 4
73	-2,918 918 919	0,081 081 081 08	4,605 405 405	-0,383 783 783 8
74	-2,92	0,08	4,610 526 316	-0,378 947 368 4
75	-2,921 052 632	0,078 947 368 42	4,615 516 062	-0,374 231 032 1
76	-2,922 077 922	0,077 922 077 92	4,620 379 620	-0,369 630 369 6
77	-2,923 076 923	0,076 923 076 92	4,625 121 713	-0,365 141 187 9
78	-2,924 050 633	0,075 949 367 08	4,629 746 835	-0,360 759 493 7
79	-2,925	0,075	4,634 259 260	-0,356 481 481 5
80	-2,925 925 926	0,074 074 074 07	4,638 663 052	-0,352 303 523 0
81	-2,926 829 268	0,073 170 731 70	4,642 962 091	-0,348 222 156 8
82	-2,927 710 843	0,072 289 156 62	4,647 160 068	-0,344 234 079 1
83	-2,928 571 428	0,071 428 571 42	4,651 260 504	-0,340 336 134 4
84	-2,929 411 765	0,070 588 235 30	4,655 266 758	-0,336 525 307 8
85	-2,930 232 558	0,069 767 441 86	4,659 182 037	-0,332 798 716 9
86	-2,931 034 483	0,068 965 517 24	4,663 009 403	-0,329 153 604 9
87	-2,931 818 182	0,068 181 818 18	4,666 751 789	-0,325 587 334 1
88	-2,932 584 270	0,067 415 730 34	4,670 411 986	-0,322 097 378 4
89	-2,933 333 333	0,066 666 666 67	4,673 992 674	-0,318 681 318 7
90	-2,934 065 934	0,065 934 065 93	4,677 496 416	-0,315 336 837 0
91	-2,934 782 608	0,065 217 391 30	4,680 925 664	-0,312 061 711 0
92	-2,935 483 871	0,064 516 129 04	4,684 282 773	-0,308 853 809 2
93	-2,936 170 213	0,063 829 787 24	4,687 569 990	-0,305 711 086 3
94	-2,936 842 105	0,063 157 894 74	4,690 789 473	-0,302 631 578 9
95	-2,937 5	0,062 5	4,693 943 301	-0,299 613 402 2
96	-2,938 144 330	0,061 855 670 10	4,697 033 453	-0,296 654 744 5
97	-2,938 775 510	0,061 224 489 80	4,700 061 841	-0,293 753 865 1
98	-2,939 393 939	0,060 606 060 61	4,703 030 303	-0,290 909 090 9
99	-2,94	0,06	4,705 940 594	-0,288 118 811 9
100	-2,940 594 060	0,059 405 940 59	4,708 794 409	-0,285 381 479 3

N	C_{22}	C_{30}	C_{31}	C_{32}
3	3			
4	2	- 0,2	0,8	- 2
5	1,428 571 429	- 0,5	1,5	- 2,5
6	1,071 428 572	- 0,833 333 333 3	2	- 2,5
7	0,833 333 333 3	- 1,166 666 667	2,333 333 333	-2,333 333 333
8	0,666 666 666 7	- 1,484 848 485	2,545 454 545	- 2,121 212 121
9	0,545 454 545 5	-1,781 818 182	2,672 727 273	- 1,909 090 909
10	0,454 545 454 5	-2,055 944 056	2,741 258 742	- 1,713 286 714
11	0,384 615 384 6	- 2,307 692 308	2,769 230 769	- 1,538 461 538
12	0,329 670 329 7	-2,538 461 538	2,769 230 769	- 1,384 615 384
13	0,285 714 285 7	- 2,75	2,75	- 1,25
14	0,25	-2,944 117 648	2,717 647 060	- 1,132 352 941
15	0,220 588 235 2	-3,122 549 020	2,676 470 588	- 1,029 411 765
16	0,196 078 431 4	-3,286 893 705	2,629 514 964	-0,939 112 487 0
17	0,175 438 596 5	-3,438 596 491	2,578 947 368	-0,859 649 122 8
18	0,157 894 736 8	-3,578 947 369	2,526 315 790	-0,789 473 684 2
19	0,142 857 142 9	-3,709 090 909	2,472 727 273	-0,727 272 727 3
20	0,129 870 129 9	-3,830 039 526	2,418 972 332	-0,671 936 758 9
21	0,118 577 075 1	-3,942 687 747	2,365 612 648	-0,622 529 644 3
22	0,108 695 652 2	-4,047 826 087	2,313 043 478	- 0,578 260 869 6
23	0,1	-4,146 153 846	2,261 538 461	- 0,538 461 538 4
24	0,092 307 692 28	-4,238 290 598	2,211 282 051	- 0,502 564 102 5
25	0,085 470 085 50	-4,324 786 325	2,162 393 163	- 0,470 085 470 1
26	0,079 365 079 38	-4,406 130 268	2,114 942 528	- 0,440 613 026 8
27	0,073 891 625 64	-4,482 758 621	2,068 965 517	- 0,413 793 103 4
28	0,068 965 517 22	-4,555 061 179	2,024 471 635	- 0,389 321 468 2
29	0,064 516 129 02	-4,623 387 098	1,981 451 613	- 0,366 935 484 0
30	0,060 483 870 96	-4,688 049 852	1,939 882 697	- 0,346 407 624 5
31	0,056 818 181 82	- 4,749 331 550	1,899 732 620	-0,327 540 106 9
32	0,053 475 935 83	-4,807 486 633	1,860 962 568	-0,310 160 427 9
33	0,050 420 168 07	- 4,862 745 098	1,823 529 412	-0,294 117 647 1
34	0,047 619 047 62	-4,915 315 315	1,787 387 387	-0,279 279 279 3
35	0,045 045 045 05	- 4,965 386 439	1,752 489 332	-0,265 528 686 6
36	0,042 674 253 20	- 5,013 130 540	1,718 787 614	-0,252 762 884 4
37	0,040 485 829 96	- 5,058 704 454	1,686 234 818	-0,240 890 688 2
38	0,038 461 538 46	-5,102 251 407	1,654 784 240	-0,229 831 144 5
39	0,036 585 365 86	-5,143 902 439	1,624 390 244	-0,219 512 195 1
40	0,034 843 205 57	-5,183 777 652	1,595 008 508	-0,209 869 540 6
41	0,033 222 591 36	-5,221 987 315	1,566 596 194	-0,200 845 666 0
42	0,031 712 473 57	-5,258 632 840	1,539 112 051	-0,192 389 006 4
43	0,030 303 030 30	-5,293 807 641	1,512 516 469	-0,184 453 227 9
44	0,028 985 507 24	-5,327 597 903	1,486 771 508	-0,176 996 608 1
45	0,027 752 081 14	-5,360 083 256	1,461 840 888	-0,169 981 498 6
46	0,026 595 744 68	-5,391 337 386	1,437 689 970	-0,163 373 860 2
47	0,025 510 204 08	-5,421 428 571	1,414 285 714	-0,157 142 857 1
48	0,024 489 795 92	-5,450 420 168	1,391 596 639	-0,151 260 504 2
49	0,023 529 411 76	-5,478 371 040	1,369 592 760	-0,145 701 357 4
50	0,022 624 434 39	-5,505 335 952	1,348 245 539	-0,140 442 243 7

N	C_{22}	C_{30}	C_{31}	C_{32}
51	0,021 770 682 15	-5,531 365 909	1,327 527 818	-0,135 462 022 3
52	0,020 964 360 59	-5,556 508 480	1,307 413 760	-0,130 741 376 0
53	0,020 202 020 20	-5,580 808 081	1,287 878 788	-0,126 262 626 3
54	0,019 480 519 48	-5,604 306 221	1,268 899 522	-0,122 009 569 4
55	0,018 796 992 48	-5,627 041 743	1,250 453 721	-0,117 967 332 1
56	0,018 148 820 33	-5,649 051 031	1,232 520 225	-0,114 122 243 1
57	0,017 533 606 08	-5,670 368 206	1,215 078 901	-0,110 461 718 3
58	0,016 949 152 54	-5,691 025 285	1,198 110 586	-0,106 974 159 5
59	0,016 393 442 62	-5,711 052 353	1,181 597 038	-0,103 648 863 0
60	0,015 864 621 89	-5,730 477 702	1,165 520 888	-0,100 475 938 7
61	0,015 360 983 10	-5,749 327 957	1,149 865 591	-0,097 446 236 56
62	0,014 880 952 38	-5,767 628 207	1,134 615 385	-0,094 551 282 08
63	0,014 423 076 92	-5,785 402 099	1,119 755 245	-0,091 783 216 80
64	0,013 986 013 99	-5,802 671 956	1,105 270 849	-0,089 134 745 86
65	0,013 568 521 03	-5,819 458 856	1,091 148 535	-0,086 599 090 12
66	0,013 169 446 88	-5,835 782 722	1,077 375 272	-0,084 169 943 11
67	0,012 787 723 79	-5,851 662 406	1,063 938 619	-0,081 841 432 26
68	0,012 422 360 25	-5,867 115 739	1,050 826 699	-0,079 608 083 30
69	0,012 072 434 61	-5,882 159 623	1,038 028 169	-0,077 464 788 72
70	0,011 737 089 20	-5,896 810 084	1,025 532 189	-0,075 406 778 57
71	0,011 415 525 11	-5,911 082 314	1,013 328 397	-0,073 429 593 96
72	0,011 106 997 41	-5,924 990 742	1,001 406 886	-0,071 529 063 28
73	0,010 810 810 81	-5,938 549 075	0,989 758 179 3	-0,069 701 280 23
74	0,010 526 315 79	-5,951 770 335	0,978 373 205 7	-0,067 942 583 73
75	0,010 252 904 99	-5,964 666 912	0,967 243 283 1	-0,066 249 539 94
76	0,009 990 009 990	-5,977 250 597	0,956 360 095 6	-0,064 618 925 38
77	0,009 737 098 344	-5,989 532 620	0,945 715 676 8	-0,063 047 711 78
78	0,009 493 670 886	-6,001 523 676	0,935 302 391 0	-0,061 533 052 04
79	0,009 259 259 259	-6,013 233 966	0,925 112 917 8	-0,060 072 267 39
80	0,009 033 423 666	-6,024 673 219	0,915 140 235 7	-0,058 662 835 62
81	0,008 815 750 806	-6,035 850 720	0,905 377 608 0	-0,057 302 380 25
82	0,008 605 851 978	-6,046 775 336	0,895 818 568 4	-0,055 988 660 52
83	0,008 403 361 344	-6,057 455 540	0,886 456 908 3	-0,054 719 562 24
84	0,008 207 934 343	-6,067 899 429	0,877 286 664 4	-0,053 493 089 29
85	0,008 019 246 192	-6,078 114 748	0,868 302 106 9	-0,052 307 355 84
86	0,007 836 990 594	-6,088 108 908	0,859 497 728 2	-0,051 160 579 06
87	0,007 660 878 450	-6,097 889 003	0,850 868 232 9	-0,050 051 072 53
88	0,007 490 636 706	-6,107 461 827	0,842 408 527 8	-0,048 977 239 99
89	0,007 326 007 326	-6,116 833 891	0,834 113 712 4	-0,047 937 569 68
90	0,007 166 746 296	-6,126 011 436	0,825 979 070 0	-0,046 930 628 98
91	0,007 012 622 718	-6,135 000 448	0,818 000 059 7	-0,045 955 059 53
92	0,006 863 417 982	-6,143 806 668	0,810 172 307 9	-0,045 009 572 66
93	0,006 718 924 974	-6,152 435 610	0,802 491 601 3	-0,044 092 945 12
94	0,006 578 947 368	-6,160 892 566	0,794 953 879 5	-0,043 204 015 19
95	0,006 443 298 972	-6,169 182 621	0,787 555 228 2	-0,042 341 678 94
96	0,006 311 803 074	-6,177 310 662	0,780 291 873 1	-0,041 504 886 87
97	0,006 184 291 896	-6,185 281 385	0,773 160 173 2	-0,040 692 640 69
98	0,006 060 606 060	-6,193 099 310	0,766 156 615 7	-0,039 903 990 40
99	0,005 940 594 059	-6,200 768 783	0,759 277 810 1	-0,039 138 031 45
100	0,005 824 111 823	-6,208 293 988	0,752 520 483 4	-0,038 393 902 22

N	C_{33}	C_{40}	C_{41}	C_{42}
4	4			
5	2,5	0,071 428 571 43	-0,357 142 857 2	1,071 428 571
6	1,666 666 667	0,214 285 714 3	-0,857 142 857 2	1,928 571 429
7	1,166 666 667	0,409 090 909 1	-1,363 636 364	2,454 545 455
8	0,848 484 848 5	0,636 363 636 4	-1,818 181 818	2,727 272 727
9	0,636 363 636 4	0,881 118 881 1	-2,202 797 203	2,832 167 832
10	0,489 510 489 6	1,132 867 133	-2,517 482 517	2,832 167 832
11	0,384 615 384 6	1,384 615 385	-2,769 230 769	2,769 230 769
12	0,307 692 307 6	1,631 868 132	-2,967 032 967	2,670 329 670
13	0,25	1,871 848 739	-3,119 747 899	2,552 521 008
14	0,205 882 353 0	2,102 941 177	-3,235 294 118	2,426 470 589
15	0,171 568 627 5	2,324 303 405	-3,320 433 436	2,298 761 610
16	0,144 478 844 2	2,535 603 715	-3,380 804 953	2,173 374 613
17	0,122 807 017 5	2,736 842 104	-3,421 052 630	2,052 631 578
18	0,105 263 157 9	2,928 229 666	-3,444 976 077	1,937 799 043
19	0,090 909 090 91	3,110 107 283	-3,455 674 759	1,829 474 872
20	0,079 051 383 40	3,282 891 022	-3,455 674 760	1,727 837 380
21	0,069 169 960 48	3,447 035 573	-3,447 035 573	1,632 806 324
22	0,060 869 565 22	3,603 010 033	-3,431 438 127	1,544 147 157
23	0,053 846 153 84	3,751 282 051	-3,410 256 410	1,461 538 461
24	0,047 863 247 86	3,892 307 693	-3,384 615 385	1,384 615 385
25	0,042 735 042 74	4,026 525 199	-3,355 437 666	1,312 997 347
26	0,038 314 176 24	4,154 351 396	-3,323 481 117	1,246 305 419
27	0,034 482 758 62	4,276 179 883	-3,289 369 141	1,184 172 891
28	0,031 145 717 46	4,392 380 423	-3,253 615 128	1,126 251 391
29	0,028 225 806 46	4,503 299 121	-3,216 642 229	1,072 214 076
30	0,025 659 824 04	4,609 259 101	-3,178 799 380	1,021 756 943
31	0,023 395 721 92	4,710 561 498	-3,140 374 332	0,974 598 930 5
32	0,021 390 374 34	4,807 486 632	-3,101 604 278	0,930 481 283 5
33	0,019 607 843 14	4,900 295 253	-3,062 684 533	0,889 166 477 4
34	0,018 018 018 02	4,989 229 831	-3,023 775 655	0,850 436 903 0
35	0,016 595 542 91	5,074 515 814	-2,985 009 302	0,814 093 446 0
36	0,015 318 962 69	5,156 362 841	-2,946 493 052	0,779 954 043 1
37	0,014 170 040 49	5,234 965 933	-2,908 314 407	0,747 852 276 1
38	0,013 133 208 26	5,310 506 567	-2,870 544 090	0,717 636 022 5
39	0,012 195 121 95	5,383 153 716	-2,833 238 798	0,689 166 194 0
40	0,011 344 299 49	5,453 064 803	-2,796 443 488	0,662 315 563 1
41	0,010 570 824 52	5,520 386 590	-2,760 193 295	0,636 967 683 5
42	0,009 866 102 890	5,585 255 997	-2,724 515 121	0,613 015 902 2
43	0,009 222 661 396	5,647 800 858	-2,689 428 980	0,590 362 459 0
44	0,008 633 980 882	5,708 140 610	-2,654 949 121	0,568 917 668 8
45	0,008 094 357 076	5,766 386 943	-2,621 084 974	0,548 599 180 6
46	0,007 598 784 194	5,822 644 377	-2,587 841 945	0,529 331 307 0
47	0,007 142 857 142	5,877 010 804	-2,555 222 089	0,511 044 417 7
48	0,006 722 689 076	5,929 577 985	-2,523 224 675	0,493 674 392 9
49	0,006 334 841 628	5,980 431 999	-2,491 846 666	0,477 162 127 6
50	0,005 976 265 688	6,029 653 662	-2,461 083 127	0,461 453 086 4

N	C_{33}	C_{40}	C_{41}	C_{42}
51	0,005 644 250 928	6,077 318 907	−2.430 927 563	0,446 496 899 3
52	0,005 336 382 694	6,123 499 143	−2,401 372 213	0,432 246 998 3
53	0,005 050 505 051	6,168 261 563	−2,372 408 293	0,418 660 287 1
54	0,004 784 688 996	6,211 669 457	−2,344 026 210	0,405 696 844 1
55	0,004 537 205 082	6,253 782 470	−2,316 215 730	0,393 319 652 2
56	0,004 306 499 738	6,294 656 864	−2,288 966 132	0,381 494 355 4
57	0,004 091 174 752	6,334 345 746	−2,262 266 338	0,370 189 037 1
58	0,003 889 969 436	6,372 899 282	−2,236 105 011	0,359 374 019 7
59	0,003 701 745 108	6,410 364 887	−2,210 470 651	0,349 021 681 7
60	0,003 525 471 532	6,446 787 416	−2,185 351 666	0,339 106 293 1
61	0,003 360 215 054	6,482 209 321	−2,160 736 440	0,329 603 863 8
62	0,003 205 128 206	6,516 670 830	−2,136 613 387	0,320 492 008 1
63	0,003 059 440 560	6,550 210 049	−2,112 970 984	0,311 749 817 2
64	0,002 922 450 684	6,582 863 143	−2,089 797 823	0,303 357 748 5
65	0,002 793 519 036	6,614 664 415	−2,067 082 630	0,295 297 518 5
66	0,002 672 061 686	6,645 646 448	−2,044 814 292	0,287 552 009 8
67	0,002 557 544 758	6,675 840 208	−2,022 981 881	0,280 105 183 5
68	0,002 449 479 480	6.705 275 129	−2,001 574 665	0,272 941 999 8
69	0,002 347 417 840	6,733 979 218	−1,980 582 123	0,266 048 344 9
70	0,002 250 948 614	6,761 979 132	−1,959 993 951	0,259 410 964 1
71	0,002 159 693 940	6,789 300 259	−1,939 800 074	0,253 017 401 0
72	0,002 073 306 182	6,815 966 796	−1,919 990 647	0,246 855 940 3
73	0,001 991 465 149	6,842 001 810	−1,900 556 058	0,240 915 556 7
74	0,001 913 875 598	6,867 427 310	−1,891 486 934	0,235 185 866 8
75	0,001 840 264 998	6,892 264 298	−1,862 774 135	0,229 657 085 1
76	0,001 770 381 517	6,916 532 835	−1,844 408 756	0,224 319 983 8
77	0,001 703 992 210	6,940 252 082	−1,826 382 127	0,219 165 855 2
78	0,001 640 881 388	6,963 440 362	−1,808 685 808	0,214 186 477 3
79	0,001 580 849 142	6,986 115 193	−1,791 311 588	0,209 374 081 7
80	0,001 523 710 016	7,008 293 336	−1,774 251 477	0,204 721 324 3
81	0,001 469 291 801	7,029 990 839	−1,757 497 710	0,200 221 258 1
82	0,001 417 434 443	7,051 223 066	−1,741 042 732	0,195 867 307 4
83	0,001 367 989 056	7,072 004 744	−1,724 879 206	0,191 653 245 1
84	0,001 320 817 020	7,092 349 982	−1,708 999 996	0,187 573 170 2
85	0,001 275 789 167	7,112 272 313	−1,693 398 170	0,183 621 488 3
86	0,001 232 785 038	7,131 784 720	−1,678 066 993	0,179 792 892 1
87	0,001 191 692 203	7,150 899 663	−1,662 999 922	0,176 082 344 6
88	0,001 152 405 647	7,169 629 101	−1,648 190 598	0,172 485 062 6
89	0,001 114 827 202	7,187 984 527	−1,633 632 847	0,168 996 501 4
90	0,001 078 865 034	7,205 976 979	−1,619 320 669	0,165 612 341 2
91	0,001 044 433 171	7,223 617 068	−1,605 248 237	0,162 328 473 4
92	0,001 011 451 071	7,240 915 001	−1,591 409 890	0,159 140 989 0
93	0,000 979 843 225 0	7,257 880 596	−1,577 800 129	0,156 046 166 7
94	0,000 949 538 795 4	7,274 523 294	−1,564 413 612	0,153 040 462 0
95	0,000 920 471 281 2	7,290 852 189	−1,551 245 147	0,150 120 498 1
96	0,000 892 578 212 2	7,306 876 040	−1,538 289 693	0,147 283 055 7
97	0,000 865 800 865 8	7,322 603 281	−1,525 542 350	0,144 525 064 8
98	0,000 840 084 008 4	7,338 042 040	−1,512 998 359	0,141 843 596 1
99	0,000 815 375 655 2	7,353 200 151	−1,500 653 092	0,139 235 853 9
100	0,000 791 626 849 8	7,368 085 172	−1,488 502 055	0,136 699 168 3

N	C_{43}	C_{44}	C_{50}	C_{51}
5	-2,5	5		
6	-3	3	- 0,023 809 523 81	0,142 857 142 9
7	-2,863 636 364	1,909 090 909	- 0,083 333 333 33	0,416 666 666 7
8	-2,545 454 545	1,272 727 273	- 0,179 487 179 5	0,769 230 769 2
9	- 2,202 797 203	0,881 118 881 1	- 0,307 692 307 7	1,153 846 154
10	-1,888 111 888	0,629 370 629 4	- 0,461 538 461 5	1,538 461 538
11	-1,615 384 615	0,461 538 461 5	- 0,634 615 384 5	1,903 846 154
12	-1,384 615 385	0,346 153 846 2	- 0,821 266 968 5	2,239 819 005
13	-1,191 176 471	0,264 705 882 4	- 1,016 806 723	2,542 016 807
14	-1,029 411 765	0,205 882 353 0	- 1,217 492 260	2,809 597 523
15	-0,893 962 848 2	0,162 538 699 7	- 1,420 407 637	3,043 730 650
16	-0,780 185 758 4	0,130 030 959 7	- 1,623 323 013	3,246 646 027
17	-0,684 210 526 1	0,105 263 157 9	- 1,824 561 403	3,421 052 631
18	-0,602 870 813 5	0,086 124 401 93	- 2,022 883 295	3,569 794 051
19	-0,533 596 837 7	0,071 146 245 03	- 2,217 391 304	3,695 652 173
20	-0,474 308 300 4	0,059 288 537 55	- 2,407 453 416	3,801 242 236
21	-0,423 320 158 1	0,049 802 371 54	- 2,592 642 141	3,888 963 211
22	-0,379 264 214 0	0,042 140 468 22	- 2,772 686 734	3,960 981 048
23	-0,341 025 641 0	0,035 897 435 90	- 2,947 435 897	4,019 230 768
24	-0,307 692 307 7	0,030 769 230 77	- 3,116 828 765	4,065 428 824
25	- 0,278 514 588 8	0,026 525 198 94	- 3,280 872 384	4,101 090 480
26	-0,252 873 563 3	0,022 988 505 75	- 3,439 624 274	4,127 549 129
27	-0,230 255 839 9	0,020 022 246 94	- 3,593 178 929	4,145 975 687
28	-0,210 233 592 9	0,017 519 466 08	- 3,741 657 397	4,157 397 108
29	-0,192 448 680 4	0,015 395 894 43	- 3,885 199 241	4,162 713 473
30	-0,176 599 965 5	0,013 584 612 73	- 4,023 956 357	4,162 713 472
31	-0,162 433 155 1	0,012 032 085 56	- 4,158 088 235	4,158 088 235
32	-0,149 732 620 3	0,010 695 187 17	- 4,287 758 346	4,149 443 561
33	-0,138 314 785 4	0,009 538 950 715	- 4,413 131 398	4,137 310 686
34	-0,128 022 759 6	0,008 534 850 639	- 4,534 371 270	4,122 155 700
35	-0,118 721 960 9	0,007 659 481 347	- 4,651 639 494	4,104 387 789
36	-0,110 296 531 4	0,006 893 533 210	- 4,765 094 116	4,084 366 385
37	-0,102 646 390 8	0,006 220 993 384	- 4,874 888 909	4,062 407 424
38	-0,095 684 803 00	0,005 628 517 824	- 4,981 172 826	4,038 788 778
39	-0,089 336 358 49	0,005 104 934 771	- 5,084 089 620	4,013 754 963
40	-0,083 535 296 24	0,004 640 849 791	- 5,183 777 652	3,987 521 270
41	-0,078 224 101 48	0,004 228 329 810	- 5,280 369 782	3,960 277 336
42	-0,073 352 330 17	0,003 860 648 956	- 5,373 993 360	3,932 190 263
43	-0,068 875 620 22	0,003 532 083 088	- 5,464 770 274	3,903 407 339
44	-0,064 754 856 61	0,003 237 742 831	- 5,552 817 057	3,874 058 412
45	-0,060 955 464 51	0,002 973 437 293	- 5,638 245 011	3,844 257 962
46	-0,057 446 808 51	0,002 735 562 310	- 5,721 160 378	3,814 106 919
47	-0,054 201 680 67	0,002 521 008 403	- 5,801 664 512	3,783 694 247
48	-0,051 195 862 96	0,002 327 084 680	- 5,879 854 060	3,753 098 336
49	-0,048 407 752 07	0,002 151 455 648	- 5,955 821 167	3,722 388 229
50	-0,045 818 036 94	0,001 992 088 563	- 6,029 653 660	3,691 624 690

N	C_{43}	C_{44}	C_{50}	C_{51}
51	-0,043 409 420 76	0,001 847 209 394	-6,101 435 253	3,660 861 152
52	-0,041 166 380 79	0,001 715 265 866	-6,171 245 726	3,630 144 544
53	-0,039 074 960 13	0,001 594 896 332	-6,239 161 120	3,599 516 031
54	-0,037 122 587 04	0,001 484 903 482	-6,305 253 929	3,569 011 658
55	-0,035 297 917 50	0,001 384 232 059	-6,369 593 254	3,538 662 919
56	-0,033 590 697 96	0,001 291 949 922	-6,432 244 994	3,508 497 269
57	-0,031 991 645 18	0,001 207 231 894	-6,493 271 986	3,478 538 564
58	-0,030 492 341 06	0,001 129 345 965	-6,552 734 183	3,448 807 465
59	-0,029 085 140 14	0,001 057 641 460	-6,610 688 791	3,419 321 788
60	-0,027 763 088 32	0,000 991 538 868 6	-6,667 190 404	3,390 096 815
61	-0,026 519 851 11	0,000 930 521 091 5	-6,722 291 149	3,361 145 575
62	-0,025 349 650 35	0,000 874 125 874 3	-6,776 040 811	3,332 479 087
63	-0,024 247 208 01	0,000 821 939 254 5	-6,828 486 947	3,304 106 587
64	-0,023 207 696 61	0,000 773 589 887 0	-6,879 675 007	3,276 035 717
65	-0,022 226 694 94	0,000 728 744 096 5	-6,929 648 433	3,248 272 703
66	-0,021 300 148 87	0,000 687 101 576 5	-6,978 448 774	3,220 822 511
67	-0,020 424 336 30	0,000 648 391 628 6	-7,026 115 774	3,193 688 988
68	-0,019 595 835 88	0,000 612 369 871 4	-7,072 687 465	3,166 874 984
69	-0,018 811 499 13	0,000 578 815 357 9	-7,118 200 254	3,140 382 465
70	-0,018 068 425 36	0,000 547 528 041 3	-7,162 689 006	3,114 212 611
71	-0,017 363 939 28	0,000 518 326 545 7	-7,206 187 117	3,088 365 907
72	-0,016 695 570 84	0,000 491 046 201 2	-7,248 726 592	3,062 842 222
73	-0,016 061 037 11	0,000 465 537 307 6	-7,290 338 111	3,037 640 880
74	-0,015 458 225 99	0,000 441 663 599 6	-7,331 051 094	3,012 760 724
75	-0,014 885 181 44	0,000 419 300 885 6	-7,370 893 763	2,988 200 174
76	-0,014 340 090 29	0,000 398 335 841 8	-7,409 893 200	2,963 957 280
77	-0,013 821 270 15	0,000 378 664 935 6	-7,448 075 405	2,940 029 765
78	-0,013 327 158 59	0,000 360 193 475 3	-7,485 465 343	2,916 415 068
79	-0,012 856 303 26	0,000 342 834 753 7	-7,522 086 993	2,893 110 382
80	-0,012 407 352 99	0,000 326 509 289 2	-7,557 963 401	2,870 112 684
81	-0,011 979 049 63	0,000 311 144 146 2	-7,593 116 719	2,847 418 770
82	-0,011 570 220 69	0,000 296 672 325 4	-7,627 568 248	2,825 025 277
83	-0,011 179 772 63	0,000 283 032 218 5	-7,661 338 474	2,802 928 710
84	-0,010 806 684 71	0,000 270 167 117 6	-7,694 447 108	2,781 125 461
85	-0,010 450 003 40	0,000 258 024 775 3	-7,726 913 130	2,759 611 832
86	-0,010 108 837 31	0,000 246 557 007 5	-7,758 754 806	2,738 384 049
87	-0,009 782 352 480	0,000 235 719 336 9	-7,789 989 730	2,717 438 278
88	-0,009 469 768 142	0,000 225 470 670 0	-7,820 634 849	2,696 770 638
89	-0,009 170 352 790	0,000 215 773 006 8	-7,850 706 503	2,676 377 217
90	-0,008 883 420 600	0,000 206 591 176 8	-7,880 220 438	2,656 254 080
91	-0,008 608 328 137	0,000 197 892 600 8	-7,909 191 835	2,636 397 278
92	-0,008 344 471 335	0,000 189 647 075 8	-7,937 635 345	2,616 802 861
93	-0,008 091 282 715	0,000 181 826 577 9	-7,965 565 098	2,597 466 879
94	-0,007 848 228 821	0,000 174 405 084 9	-7,992 994 727	2,578 385 396
95	-0,007 614 807 873	0,000 167 358 414 8	-8,019 937 409	2,559 554 492
96	-0,007 390 547 597	0,000 160 664 078 2	-8,046 405 846	2,540 970 267
97	-0,007 175 003 215	0,000 154 301 144 4	-8,072 412 329	2,522 628 853
98	-0,006 967 755 600	0,000 148 250 119 2	-8,097 968 722	2,504 526 409
99	-0,006 768 409 564	0,000 142 492 832 9	-8,123 086 494	2,486 659 131
100	-0,006 570 592 290	0,000 137 012 339 4	-8,147 776 722	2,469 023 249

N	C_{52}	C_{53}	C_{54}	C_{55}
6	-0,5	1,333 333 333	-3	6
7	-1,166 666 667	2,333 333 333	-3,5	3,5
8	-1,794 871 795	2,871 794 872	-3,230 769 230	2,153 846 154
9	-2,307 692 308	3,076 923 077	-2,769 230 769	1,384 615 385
10	-2,692 307 692	3,076 923 077	-2,307 692 308	0,923 076 923 1
11	-2,961 538 461	2,961 538 461	-1,903 846 154	0,634 615 384 5
12	-3,135 746 607	2,787 330 317	-1,567 873 304	0,447 963 801 0
13	-3,235 294 117	2,588 235 294	-1,294 117 647	0,323 529 411 7
14	-3,277 863 777	2,383 900 929	-1,072 755 418	0,238 390 092 9
15	-3,277 863 777	2,185 242 518	-0,893 962 848 4	0,178 792 569 7
16	-3,246 646 027	1,997 936 016	-0,749 226 006 1	0,136 222 910 2
17	-3,192 982 456	1,824 561 403	-0,631 578 947 3	0,105 263 157 9
18	-3,123 569 794	1,665 903 890	-0,535 469 107 6	0,082 379 862 71
19	-3,043 478 260	1,521 739 130	-0,456 521 739 1	0,065 217 391 29
20	-2,956 521 740	1,391 304 348	-0,391 304 347 9	0,052 173 913 05
21	-2,865 551 840	1,273 578 596	-0,337 123 745 9	0,042 140 468 24
22	-2,772 686 734	1,167 447 046	-0,291 861 761 4	0,034 336 677 82
23	-2,679 487 179	1,071 794 872	-0,253 846 153 8	0,028 205 128 20
24	-2,587 091 070	0,985 558 502 8	-0,221 750 663 1	0,023 342 175 07
25	-2,496 315 945	0,907 751 252 6	-0,194 518 125 5	0,019 451 812 55
26	-2,407 736 992	0,837 473 736 3	-0,171 301 446 1	0.016 314 423 43
27	-2,321 746 385	0,773 915 461 6	-0,151 418 242 5	0,013 765 294 77
28	-2,238 598 443	0,716 351 501 7	-0,134 315 906 6	0,011 679 644 05
29	-2,158 444 023	0,664 136 622 4	-0,119 544 592 0	0,009 962 049 336
30	-2,081 356 736	0,616 698 292 2	-0,106 736 242 9	0,008 538 899 431
31	-2,007 352 941	0,573 529 411 7	-0,095 588 235 29	0,007 352 941 176
32	-1,936 406 995	0,534 181 240 0	-0,085 850 566 43	0,006 359 300 476
33	-1,868 462 890	0.498 256 770 8	-0,077 315 705 81	0,005 522 550 415
34	-1,803 443 119	0,465 404 675 9	-0,069 810 701 38	0,004 814 531 130
35	-1,741 255 425	0,435 313 856 4	-0,063 190 721 08	0,004 212 714 739
36	-1,681 797 923	0,407 708 587 5	-0,057 334 020 11	0,003 698 969 040
37	-1,624 962 970	0,382 344 228 2	-0,052 137 849 29	0,003 258 615 581
38	-1.570 640 080	0,359 003 446 9	-0,047 515 162 09	0,002 879 706 793
39	-1,518 718 094	0,337 492 909 8	-0,043 391 945 55	0,002 552 467 385
40	-1,469 086 784	0,317 640 385 7	-0,039 705 048 21	0,002 268 859 898
41	-1,421 638 018	0,299 292 214 3	-0,036 400 404 45	0,002 022 244 692
42	-1,376 266 592	0,282 311 095 8	-0,033 431 577 14	0,001 807 112 278
43	-1,332 870 799	0,266 574 159 7	-0,030 758 556 89	0,001 618 871 415
44	-1,291 352 804	0,251 971 278 8	-0,028 346 768 87	0,001 453 680 455
45	-1,251 618 871	0,238 403 594 5	-0,026 166 248 18	0,001 308 312 409
46	-1,213 579 474	0,225 782 227 8	-0,024 190 952 97	0,001 180 046 487
47	-1,177 149 321	0,214 027 149 3	-0,022 398 190 05	0,001 066 580 478
48	-1,142 247 320	0,203 066 190 2	-0,020 768 133 09	0,000 965 959 678 4
49	-1,108 796 494	0,192 834 172 8	-0,019 283 417 28	0.000 876 518 967 5
50	-1,076 723 868	0.183 272 147 7	-0,017 928 797 06	0,000 796 835 424 9

N	C_{52}	C_{53}	C_{54}	C_{55}
51	-1,045 960 329	0,174 326 721 5	- 0,016 690 856 31	0,000 725 689 405 0
52	-1,016 440 472	0,165 949 464 9	- 0,015 557 762 33	0,000 662 032 439 7
53	-0,988 102 439 9	0,158 096 390 4	- 0,014 519 056 26	0,000 604 960 677 5
54	-0,960 887 754 1	0,150 727 490 8	- 0,013 565 474 18	0,000 553 692 823 5
55	-0,934 741 148 4	0,143 806 330 5	- 0,012 688 793 87	0,000 507 551 754 8
56	-0,909 610 403 2	0,137 299 683 5	- 0,011 881 703 38	0,000 465 949 152 1
57	-0,885 446 179 9	0,131 177 211 8	- 0,011 137 687 80	0,000 428 372 607 6
58	-0,862 201 866 2	0,125 411 180 5	- 0,010 450 931 71	0,000 394 374 781 6
59	-0,839 833 421 7	0,119 976 203 1	- 0,009 816 234 799	0,000 363 564 251 8
60	-0,818 299 231 3	0,114 849 014 9	- 0,009 228 938 699	0,000 335 597 770 9
61	-0,797 559 966 9	0,110 008 271 3	- 0,008 684 863 523	0,000 310 173 697 3
62	-0,777 578 453 7	0,105 434 366 6	- 0,008 180 252 581	0,000 287 026 406 4
63	-0,758 319 544 6	0,101 109 272 6	- 0,007 711 724 182	0,000 265 921 523 5
64	-0,739 750 000 7	0,097 016 393 54	- 0,007 276 229 515	0,000 246 651 848 0
65	-0,721 838 378 4	0,093 140 435 93	- 0,006 871 015 765	0,000 229 033 858 8
66	-0,704 554 924 3	0,089 467 291 98	- 0,006 493 593 772	0,000 212 904 713 9
67	-0,687 871 474 4	0,085 983 934 30	- 0,006 141 709 592	0,000 198 119 664 3
68	-0,671 761 360 3	0,082 678 321 27	- 0,005 813 319 465	0,000 184 549 824 3
69	-0,656 199 321 0	0,079 539 311 64	- 0,005 506 567 729	0,000 172 080 241 5
70	-0,641 161 419 9	0,076 556 587 46	- 0,005 219 767 327	0,000 160 608 225 4
71	-0,626 624 966 7	0,073 720 584 32	- 0,004 951 382 529	0,000 150 041 894 8
72	-0,612 568 444 4	0,071 022 428 34	- 0,004 700 013 640	0,000 140 298 914 6
73	-0,598 971 441 0	0,068 453 878 98	- 0,004 464 383 411	0,000 131 305 394 5
74	-0,585 814 585 2	0,066 007 277 20	- 0,004 243 324 963	0,000 122 994 926 5
75	-0,573 079 485 4	0,063 675 498 38	- 0,004 035 771 024	0,000 115 307 743 5
76	-0,560 748 674 6	0,061 451 909 55	- 0,003 840 744 347	0,000 108 189 981 6
77	-0,548 805 556 2	0,059 330 330 40	- 0,003 657 349 134	0,000 101 593 031 5
78	-0,537 234 354 8	0,057 304 997 84	- 0,003 484 763 382	0,000 095 472 969 38
79	-0,526 020 069 5	0,055 370 533 63	- 0,003 322 232 018	0,000 089 790 054 53
80	-0,515 148 430 5	0,053 521 914 85	- 0,003 169 060 748	0,000 084 508 286 61
81	-0,504 605 858 0	0,051 754 446 97	- 0,003 024 610 537	0,000 079 595 014 14
82	-0,494 379 423 5	0,050 063 739 09	- 0,002 888 292 640	0,000 075 020 588 04
83	-0,484 456 814 1	0,048 445 681 41	- 0,002 759 564 131	0,000 070 758 054 64
84	-0,474 826 298 2	0,046 896 424 51	- 0,002 637 923 879	0,000 066 782 883 01
85	-0,465 476 694 6	0,045 412 360 45	- 0,002 522 908 914	0,000 063 072 722 84
86	-0,456 397 341 5	0,043 990 105 21	- 0,002 414 091 139	0,000 059 607 188 63
87	-0,447 578 069 3	0,042 626 482 79	- 0,002 311 074 368	0,000 056 367 667 52
88	-0,439 009 173 6	0,041 318 510 45	- 0,002 213 491 631	0,000 053 337 147 74
89	-0,430 681 391 3	0,040 063 385 23	- 0,002 121 002 748	0,000 050 500 065 42
90	-0,422 585 876 4	0,038 858 471 39	- 0,002 033 292 108	0,000 047 842 167 24
91	-0,414 714 178 6	0,037 701 288 96	- 0,001 950 066 671	0,000 045 350 387 69
92	-0,407 058 222 8	0,036 589 503 18	- 0,001 871 054 140	0,000 043 012 738 84
93	-0,399 610 289 2	0,035 520 914 60	- 0,001 796 001 300	0,000 040 818 211 36
94	-0,392 362 995 0	0,034 493 450 11	- 0,001 724 672 506	0,000 038 756 685 52
95	-0,385 309 278 4	0,033 505 154 64	- 0,001 656 848 306	0,000 036 818 851 26
96	-0,378 442 380 2	0,032 554 183 25	- 0,001 592 324 180	0,000 034 996 135 83
97	-0,371 755 831 0	0,031 638 794 12	- 0,001 530 909 393	0,000 033 280 638 98
98	-0,365 243 434 6	0,030 757 341 87	- 0,001 472 425 940	0,000 031 665 073 99
99	-0,358 899 256 0	0,029 908 271 33	- 0,001 416 707 589	0,000 030 142 714 67
100	-0,352 717 607 0	0,029 090 111 92	- 0,001 363 598 996	0,000 028 707 347 29

N	C_{60}	C_{61}	C_{62}	C_{63}
7	0,007 575 757 576	-0,053 030 303 03	0,212 121 212 1	- 0,636 363 636 4
8	0,030 303 030 303	- 0,181 818 181 8	0,606 060 606 1	- 1,454 545 455
9	0,072 727 272 73	- 0,381 818 181 8	1,090 909 091	- 2,181 818 182
10	0,136 363 636 4	- 0,636 363 636 4	1,590 909 091	- 2,727 272 727
11	0,220 588 235 3	- 0,926 470 588 2	2,058 823 529	- 3,088 235 294
12	0,323 529 411 7	- 1,235 294 118	2,470 588 235	- 3,294 117 647
13	0,442 724 458 2	- 1,549 535 604	2,817 337 461	- 3,380 804 953
14	0,575 541 795 6	- 1,859 442 724	3,099 071 207	- 3,380 804 953
15	0,719 427 244 5	- 2,158 281 734	3,320 433 436	- 3,320 433 436
16	0,872 033 023 9	- 2,441 692 467	3,488 132 095	- 3,219 814 242
17	1,031 273 836	- 2,707 093 821	3,609 458 427	- 3,093 821 509
18	1,195 340 129	- 2,953 193 260	3,691 491 575	- 2,953 193 260
19	1,362 687 747	-3,179 604 743	3,740 711 462	- 2,805 533 597
20	1,532 015 810	- 3,386 561 265	3,762 845 850	- 2,656 126 482
21	1,702 239 789	- 3,574 703 557	3,762 845 850	- 2,508 563 900
22	1,872 463 768	-3,744 927 537	3,744 927 537	- 2,365 217 392
23	2,041 954 023	- 3,898 275 863	3,712 643 679	- 2,227 586 207
24	2,210 114 943	-4,035 862 069	3,668 965 518	- 2,096 551 724
25	2,376 467 681	- 4,158 818 441	3,616 363 862	- 1,972 562 106
26	2,540 631 565	- 4,268 261 030	3,556 884 191	- 1,855 765 665
27	2,702 308 120	- 4,365 266 963	3,492 213 570	- 1,746 106 785
28	2,861 267 421	-4,450 860 433	3,423 738 795	- 1,643 394 621
29	3,017 336 553	- 4,526 004 830	3,352 596 170	- 1,547 352 079
30	3,170 389 857	-4,591 599 103	3,279 713 645	- 1,457 650 509
31	3,320 340 729	-4,648 477 020	3,205 846 220	- 1,373 934 095
32	3,467 134 739	-4,697 408 357	3,131 605 571	- 1,295 836 788
33	3,610 743 870	-4,739 101 330	3,057 484 729	- 1,222 993 892
34	3,751 161 688	-4,774 205 785	2,983 878 615	- 1,155 049 787
35	3,888 399 310	-4,803 316 795	2,911 101 088	- 1,091 662 908
36	4,022 482 045	-4,826 978 454	2,839 399 091	- 1,032 508 760 2
37	4,153 446 577	-4,845 687 673	2,768 964 385	- 0,977 281 547 6
38	4,281 338 627	- 4,859 897 901	2,699 943 279	- 0,925 694 838 4
39	4,406 211 004	-4,870 022 689	2,632 444 697	- 0,877 481 565 5
40	4,528 121 979	-4,876 439 055	2,566 546 871	- 0,832 393 579 7
41	4,647 133 947	-4,879 490 645	2,502 302 895	- 0,790 200 914 1
42	4,763 312 297	- 4,879 490 645	2,439 745 323	- 0,750 690 868 5
43	4,876 724 494	- 4,876 724 494	2,378 889 997	- 0,713 666 999 1
44	4,987 439 320	-4,871 452 359	2,319 739 218	- 0,678 948 063 9
45	5,095 526 240	-4,863 911 411	2,262 284 377	- 0,646 366 964 9
46	5,201 054 890	-4,854 317 898	2,206 508 135	- 0,615 769 712 2
47	5,304 094 656	-4,842 869 034	2,152 386 237	- 0,587 014 428 4
48	5,404 714 339	-4,829 744 728	2,099 889 012	- 0,559 970 403 3
49	5,502 981 872	-4,815 109 138	2,048 982 612	- 0,534 517 203 1
50	5,598 964 115	-4,799 112 098	1,999 630 041	- 0,510 543 840 2

N	C_{60}	C_{61}	C_{62}	C_{63}
51	5,692 726 671	- 4,781 890 403	1,951 792 001	- 0,487 948 000 3
52	5,784 333 767	- 4,763 568 985	1,905 427 594	- 0,466 635 329 1
53	5,873 848 142	- 4,744 261 961	1,860 494 887	- 0,446 518 772 8
54	5,961 330 988	- 4,724 073 614	1,816 951 390	- 0,427 517 974 1
55	6,046 841 882	- 4,703 099 242	1,774 754 431	- 0,409 558 714 8
56	6,130 438 777	- 4,681 425 975	1,733 861 472	- 0,392 572 408 8
57	6,212 177 960	- 4,659 133 470	1,694 230 353	- 0,376 495 634 0
58	6,292 114 073	- 4,636 294 580	1,655 819 493	- 0,361 269 707 6
59	6,370 300 108	- 4,612 975 940	1,618 588 049	- 0,346 840 296 2
60	6,446 787 415	- 4,589 238 499	1,582 496 034	- 0,333 157 059 8
61	6,521 625 741	- 4,565 138 018	1,547 504 413	- 0,320 173 326 8
62	6,594 863 253	- 4,540 725 519	1,513 575 173	- 0,307 845 797 9
63	6,666 546 550	- 4,516 047 663	1,480 671 365	- 0,296 134 273 0
64	6,736 720 721	- 4,491 147 147	1,448 757 144	- 0,285 001 405 4
65	6,805 429 383	- 4,466 063 032	1,417 797 788	- 0,274 412 475 1
66	6,872 714 701	- 4,440 831 038	1,387 759 699	- 0,264 335 180 8
67	6,938 617 445	- 4,415 483 829	1,358 610 409	- 0,254 739 451 7
68	7,003 177 023	- 4,390 051 268	1,330 318 566	- 0,245 597 273 7
69	7,066 431 525	- 4,364 560 648	1,302 853 925	- 0,236 882 531 8
70	7,128 417 767	- 4,339 036 901	1,276 187 324	- 0,228 570 864 0
71	7,189 171 328	- 4,313 502 797	1,250 290 666	- 0,220 639 529 2
72	7,248 726 593	- 4,287 979 111	1,225 136 889	- 0,213 067 285 0
73	7,307 116 796	- 4,262 484 798	1,200 699 943	- 0,205 834 275 9
74	7,364 374 054	- 4,237 037 127	1,176 954 757	- 0,198 921 930 8
75	7,420 529 411	- 4,211 651 828	1,153 877 213	- 0,192 312 868 9
76	7,475 612 874	- 4,186 343 210	1,131 444 111	- 0,185 990 812 7
77	7,529 653 449	- 4,161 124 275	1,109 633 140	- 0,179 940 509 2
78	7,582 679 179	- 4,136 006 825	1,088 422 849	- 0,174 147 655 8
79	7,634 717 172	- 4,111 001 554	1,067 792 612	- 0,168 598 833 4
80	7,685 793 649	- 4,086 118 142	1,047 722 601	- 0,163 281 444 2
81	7,735 933 963	- 4,061 365 331	1,028 193 755	- 0,158 183 654 6
82	7,785 162 634	- 4,036 750 995	1,009 187 749	- 0,153 294 341 6
83	7,833 503 382	- 4,012 282 220	0,990 686 967 9	- 0,148 603 045 2
84	7,880 979 159	- 3,987 965 357	0,972 674 477 4	- 0,144 099 922 6
85	7,927 612 174	- 3,963 806 087	0,955 133 996 9	- 0,139 775 706 9
86	7,973 423 909	- 3,939 809 461	0,938 049 871 7	- 0,135 621 668 2
87	8,018 435 176	- 3,915 979 969	0,921 407 051 6	- 0,131 629 578 8
88	8,062 666 103	- 3,892 321 567	0,905 191 062 1	- 0,127 791 679 4
89	8,106 136 191	- 3,868 837 728	0,889 387 983 4	- 0,124 100 648 8
90	8,148 864 315	- 3,845 531 475	0,873 984 426 1	- 0,120 549 576 0
91	8,190 868 771	- 3,822 405 427	0,858 967 511 6	- 0,117 131 933 4
92	8,232 167 271	- 3,799 461 817	0,844 324 848 3	- 0,113 841 552 6
93	8,272 776 972	- 3,776 702 531	0,830 044 512 3	- 0,110 672 601 6
94	8,312 714 516	- 3,754 129 136	0,816 115 029 6	- 0,107 619 564 3
95	8,351 996 021	- 3,731 742 903	0,802 525 355 5	- 0,104 677 220 3
96	8,390 637 115	- 3,709 544 830	0,789 264 857 4	- 0,101 840 626 8
97	8,428 652 946	- 3,687 535 664	0,776 323 297 6	- 0,099 105 101 83
98	8,466 058 210	- 3,665 715 926	0,763 690 817 9	- 0,096 466 208 58
99	8,502 867 159	- 3,644 085 925	0,751 357 922 8	- 0,093 919 740 34
100	8,539 093 617	- 3,622 645 777	0,739 315 464 7	- 0,091 461 706 97

N	C_{64}	C_{65}	C_{66}	C_{70}
7	1,590 909 091	- 3,5	7,0	
8	2,727 272 727	- 4,0	4,0	- 0,002 331 002 331
9	3,272 727 273	- 3,6	2,4	- 0,010 489 510 49
10	3,409 090 909	- 3,0	1,5	- 0,027 766 351 30
11	3,308 823 529	- 2,426 470 588	0,970 588 235 2	- 0,056 561 085 97
12	3,088 235 294	- 1,941 176 470	0,647 058 823 5	- 0,098 237 675 65
13	2,817 337 461	- 1,549 535 604	0,442 724 458 2	- 0,153 250 774 0
14	2,535 603 715	- 1,239 628 483	0,309 907 120 7	- 0,221 362 229 1
15	2,263 931 888	- 0,996 130 030 9	0,221 362 229 1	- 0.301 857 585 2
16	2,012 383 901	- 0,804 953 560 5	0,160 990 712 1	- 0,393 727 285 0
17	1,784 897 025	- 0,654 462 242 3	0,118 993 135 0	- 0,495 804 729 2
18	1,582 067 818	- 0,535 469 107 6	0,089 244 851 26	- 0,606 864 988 6
19	1,402 766 798	- 0,440 869 565 2	0,067 826 086 95	- 0,725 691 699 5
20	1,245 059 289	- 0,365 217 391 3	0,052 173 913 05	- 0,851 119 894 6
21	1,106 719 368	- 0,304 347 826 1	0,040 579 710 14	- 0,982 061 416 9
22	0,985 507 246 5	- 0,255 072 463 8	0,031 884 057 98	- 1,117 518 164
23	0,879 310 344 9	- 0,214 942 528 8	0,025 287 356 33	- 1,256 587 091
24	0,780 206 896 7	- 0,182 068 965 5	0,020 229 885 06	- 1,398 459 827
25	0,704 486 466 6	- 0,154 987 022 6	0,016 314 423 44	- 1,542 418 927
26	0,632 647 385 8	- 0,132 554 690 4	0,013 255 469 04	- 1,687 832 159
27	0,569 382 647 3	- 0,113 876 529 5	0,010 845 383 76	- 1,834 145 783
28	0,513 560 819 2	- 0,098 246 417 59	0,008 931 492 508	- 1,980 877 446
29	0,464 205 623 6	- 0,085 104 364 32	0,007 400 379 507	- 2,127 609 108
30	0,420 476 108 3	- 0,074 003 795 07	0,006 166 982 922	- 2,273 980 250
31	0,381 648 359 6	- 0,064 586 645 47	0,005 166 931 638	- 2,419 681 502
32	0,347 099 139 7	- 0,056 564 304 24	0,004 351 100 326	- 2,564 448 772
33	0,316 291 523 7	- 0,049 702 953 72	0,003 681 700 276	- 2,708 057 903
34	0,288 762 446 6	- 0,043 812 233 28	0,003 129 445 235	- 2,850 319 857
35	0,264 111 993 9	- 0,038 736 425 77	0,002 671 477 639	- 2,991 076 393
36	0,241 994 240 7	- 0,034 347 569 64	0,002 289 837 976	- 3,130 196 225
37	0,222 109 442 6	- 0,030 540 048 36	0,001 970 325 701	- 3,267 571 608
38	0,204 197 390 8	- 0,027 226 318 78	0,001 701 644 923	- 3,403 115 319
39	0,188 031 764 0	- 0,024 333 522 41	0,001 474 758 934	- 3,536 757 996
40	0,173 415 329 1	- 0,021 800 784 23	0,001 282 399 072	- 3,668 445 794
41	0,160 175 861 0	- 0,019 577 049 67	0,001 118 688 553	- 3,798 138 322
42	0,148 162 671 4	- 0,017 619 344 71	0,000 978 852 483 8	- 3,925 806 837
43	0,137 243 653 7	- 0,015 891 370 42	0,000 858 992 995 9	- 4,051 432 657
44	0,127 302 762 0	- 0,014 362 362 89	0,000 755 913 836 4	- 4,175 005 764
45	0,118 237 859 4	- 0,013 006 164 54	0,000 666 982 796 8	- 4,296 523 605
46	0,109 958 877 2	- 0,011 800 464 87	0,000 590 023 243 4	- 4,415 990 001
47	0,102 386 237 5	- 0,010 726 177 26	0,000 523 228 159 1	- 4,533 414 237
48	0,095 449 500 56	- 0,009 766 925 638	0,000 465 091 697 1	- 4,648 810 235
49	0,089 086 200 52	- 0,008 908 620 052	0,000 414 354 421 0	- 4,762 195 851
50	0,083 240 843 52	- 0,008 139 104 699	0.000 369 959 304 5	- 4,873 592 245

N	C_{64}	C_{65}	C_{66}	C_{70}
51	0,077 864 042 61	-0,007 447 864 945	0,000 331 016 219 8	-4,983 023 346
52	0,072 911 770 17	-0,006 825 782 740	0,000 296 773 162 6	-5,090 515 375
53	0,068 344 710 13	-0,006 264 931 761	0,000 266 592 840 9	-5,196 096 435
54	0,064 127 696 11	-0,005 758 405 365	0,000 239 933 556 9	-5,299 796 149
55	0,060 229 222 77	-0,005 300 171 603	0,000 216 333 534 8	-5,401 645 355
56	0,056 621 020 50	-0,004 884 950 788	0,000 195 398 031 5	-5,501 675 824
57	0,053 277 684 05	-0,004 508 111 727	0,000 176 788 695 2	-5,599 920 035
58	0,050 176 348 27	-0,004 165 583 630	0,000 160 214 755 0	-5,696 410 966
59	0,047 296 404 03	-0,003 853 781 069	0,000 145 425 700 7	-5,791 181 915
60	0,044 619 249 08	-0,003 569 539 927	0,000 132 205 182 5	-5,884 266 354
61	0,042 128 069 32	-0,003 310 062 589	0,000 120 365 912 3	-5,975 697 797
62	0,039 807 646 28	-0,003 072 870 941	0,000 109 745 390 7	-6,065 509 681
63	0,037 644 187 24	-0,002 855 765 929	0,000 100 202 313 3	-6,153 735 274
64	0,035 625 175 68	-0,002 656 792 763	0,000 091 613 543 54	-6,240 407 602
65	0,033 739 238 74	-0,002 474 210 841	0,000 083 871 553 94	-6,325 559 363
66	0,031 976 029 94	-0,002 306 467 733	0,000 076 882 257 78	-6,409 222 878
67	0,030 326 125 20	-0,002 152 176 627	0,000 070 563 168 10	-6,491 430 043
68	0,028 780 930 52	-0,002 010 096 735	0,000 064 841 830 15	-6,572 212 283
69	0,027 332 599 82	-0,001 879 116 238	0,000 059 654 483 73	-6,651 600 525
70	0,025 973 961 82	-0,001 758 237 415	0,000 054 944 919 23	-6,729 625 164
71	0,024 698 454 77	-0,001 646 563 651	0,000 050 663 496 95	-6,806 316 049
72	0,023 500 068 20	-0,001 543 288 061	0,000 046 766 304 88	-6,881 702 461
73	0,022 373 290 86	-0,001 447 683 526	0,000 043 214 433 63	-6,955 813 102
74	0,021 313 064 02	-0,001 359 093 937	0,000 039 973 351 10	-7,028 676 091
75	0,020 314 739 67	-0,001 276 926 493	0,000 037 012 362 13	-7,100 318 949
76	0,019 374 042 99	-0,001 200 644 918	0,000 034 304 140 51	-7,170 768 605
77	0,018 487 038 61	-0,001 129 763 471	0,000 031 824 323 12	-7,240 051 394
78	0,017 650 100 25	-0,001 063 841 659	0,000 029 551 157 19	-7,308 193 054
79	0,016 859 883 34	-0,001 002 479 550	0,000 027 465 193 15	-7,375 218 737
80	0,016 113 300 42	-0,000 945 313 624 6	0,000 025 549 016 88	-7,441 153 002
81	0,015 407 498 82	-0,000 892 013 089 6	0,000 023 787 015 72	-7,506 019 842
82	0,014 739 840 54	-0,000 842 276 602 2	0,000 022 165 173 74	-7,569 842 663
83	0,014 107 884 04	-0,000 795 829 355 9	0,000 020 670 892 36	-7,632 644 320
84	0,013 509 367 74	-0,000 752 420 481 8	0,000 019 292 832 87	-7,694 447 109
85	0,012 942 195 08	-0,000 711 820 729 4	0,000 018 020 777 96	-7,755 272 780
86	0,012 404 420 87	-0,000 673 820 393 0	0,000 016 845 509 83	-7,815 142 541
87	0,011 894 239 05	-0,000 638 227 461 1	0,000 015 758 702 74	-7,874 077 094
88	0,011 409 971 37	-0,000 604 865 952 2	0,000 014 752 828 10	-7,932 096 611
89	0,010 950 057 25	-0,000 573 574 427 4	0,000 013 821 070 54	-7,989 220 756
90	0,010 513 044 42	-0,000 544 204 652 3	0,000 012 957 253 63	-8,045 468 731
91	0,010 097 580 47	-0,000 516 620 395 9	0,000 012 155 774 02	-8,100 859 228
92	0,009 702 405 050	-0,000 490 696 347 4	0,000 011 411 542 96	-8,155 410 463
93	0,009 326 342 835	-0,000 466 317 141 7	0,000 010 719 934 29	-8,209 140 228
94	0,008 968 297 029	-0,000 443 376 482 3	0,000 010 076 738 23	-8,262 065 838
95	0,008 627 243 429	-0,000 421 776 345 4	0,000 009 478 120 122	-8,314 204 185
96	0,008 302 225 008	-0,000 401 426 264 1	0,000 008 920 583 647	-8,365 571 731
97	0,007 992 346 922	-0,000 382 242 678 9	0,000 008 400 937 997	-8,416 184 525
98	0,007 696 771 961	-0,000 364 148 350 9	0,000 007 916 268 497	-8,466 058 210
99	0,007 414 716 343	-0,000 347 071 828 8	0,000 007 463 910 297	-8,515 208 040
100	0,007 145 445 857	-0,000 330 946 966 0	0,000 007 041 424 809	-8,563 648 883

N	C_{71}	C_{72}	C_{73}	C_{74}
8	0,018 648 018 65	- 0,083 916 083 92	0,279 720 279 7	- 0,769 230 769 2
9	0,073 426 573 42	- 0,283 216 783 2	0,786 713 286 6	- 1,730 769 231
10	0,172 768 408 1	- 0,583 093 377 2	1,388 317 565	- 2,545 248 869
11	0,316 742 081 4	- 0,950 226 244 3	1,979 638 009	- 3,110 859 728
12	0,500 119 076 0	- 1,350 321 505	2,500 595 380	- 3,438 318 648
13	0,715 170 278 6	- 1,755 417 957	2,925 696 594	- 3,575 851 393
14	0,953 560 371 5	- 2,145 510 836	3,250 773 994	- 3,575 851 393
15	1,207 430 341	- 2,507 739 938	3,482 972 136	- 3,482 972 136
16	1,469 915 197	- 2,834 836 452	3,634 405 707	- 3,331 538 565
17	1,735 316 552	- 3,123 569 794	3,718 535 469	- 3,146 453 089
18	1,999 084 668	- 3,373 455 378	3,748 283 753	- 2,945 080 092
19	2,257 707 510	- 3,585 770 751	3,735 177 865	- 2,739 130 434
20	2,508 563 900	- 3,762 845 850	3,689 064 559	- 2,536 231 884
21	2,749 771 967	- 3,907 570 690	3,618 121 010	- 2,341 137 124
22	2,980 048 437	- 4,023 065 390	3,529 004 729	- 2,156 614 001
23	3,198 585 323	- 4,112 466 843	3,427 055 703	- 1,984 084 881
24	3,404 945 666	- 4,178 796 954	3,316 505 519	- 1,824 078 035
25	3,598 977 496	- 4,224 886 626	3,200 671 687	- 1,676 542 312
26	3,780 744 036	- 4,253 337 041	3,082 128 290	- 1,541 064 145
27	3,950 467 841	- 4,266 505 268	2,962 850 880	- 1,417 015 638
28	4,108 486 555	- 4,266 505 268	2,844 336 845	- 1,303 654 388
29	4,255 218 217	- 4,255 218 217	2,727 703 985	- 1,200 189 753
30	4,391 134 276	- 4,234 308 052	2,613 770 403	- 1,105 825 940
31	4,516 738 804	- 4,205 239 577	2,503 118 796	- 1,019 789 139
32	4,632 552 620	- 4,169 297 358	2,396 147 907	-0,941 343 820 6
33	4,739 101 330	- 4,127 604 384	2,293 113 547	-0,869 801 690 2
34	4,836 906 423	- 4,081 139 795	2,194 161 180	- 0,804 525 766 0
35	4,926 478 765	- 4,030 755 353	2,099 351 746	- 0,744 931 264 8
36	5,008 313 960	- 3,977 190 498	2,008 682 070	-0,690 484 461 4
37	5,082 889 168	- 3,921 085 930	1,922 100 946	-0,640 700 315 3
38	5,150 661 023	- 3,862 995 767	1,839 521 794	- 0,595 139 404 0
39	5,212 064 416	- 3,803 398 357	1,760 832 573	- 0,553 404 522 9
40	5,267 511 909	- 3,742 705 830	1,685 903 527	- 0,515 137 188 9
41	5,317 393 651	- 3,681 272 528	1,614 593 214	- 0,480 014 198 8
42	5,362 077 631	- 3,619 402 401	1,546 753 163	- 0,447 744 336 6
43	5,401 910 209	- 3,557 355 503	1,482 231 460	- 0,418 065 283 5
44	5,437 216 809	- 3,495 353 663	1,420 875 473	- 0,390 740 755 0
45	5,468 302 770	- 3,433 585 460	1,362 533 913	- 0,365 557 879 0
46	5,495 454 224	- 3,372 210 546	1,307 058 351	- 0,342 324 806 3
47	5,518 939 071	- 3,311 363 442	1,254 304 334	- 0,320 868 550 6
48	5,539 007 940	- 3,251 156 834	1,204 132 161	- 0,301 033 040 2
49	5,555 895 159	- 3,191 684 453	1,156 407 411	- 0,282 677 367 0
50	5,569 819 708	- 3,133 023 586	1,111 001 272	- 0,265 674 217 1

N	C_{71}	C_{72}	C_{73}	C_{74}
51	5,580 986 148	− 3,075 237 265	1,067 790 717	− 0,249 908 465 7
52	5,589 585 510	− 3,018 376 176	1,026 658 563	− 0,235 275 920 7
53	5,595 796 161	− 2,962 480 320	0,987 493 440 1	− 0,221 682 200 8
54	5,599 784 610	− 2,907 580 471	0,950 189 696 3	− 0,209 041 733 2
55	5,601 706 294	− 2,853 699 433	0,914 647 254 1	− 0,197 276 858 7
56	5,601 706 294	− 2,800 853 147	0,880 771 429 8	− 0,186 317 033 2
57	5,599 920 035	− 2,749 051 653	0,848 472 732 5	− 0,176 098 114 3
58	5,596 473 931	− 2,698 299 931	0,817 666 645 8	− 0,166 561 724 1
59	5,591 485 987	− 2,648 598 625	0,788 273 400 4	− 0,157 654 680 1
60	5,585 066 369	− 2,599 944 689	0,760 217 745 4	− 0,149 328 485 7
61	5,577 317 944	− 2,552 331 941	0,733 428 718 5	− 0,141 538 875 5
62	5,568 336 756	− 2,505 751 540	0,707 839 418 1	− 0,134 245 406 9
63	5,558 212 506	− 2,460 192 421	0,683 386 783 5	− 0,127 411 095 2
64	5,547 028 980	− 2,415 641 652	0,660 011 380 4	− 0,121 002 086 4
65	5,534 864 443	− 2,372 084 761	0,637 657 193 9	− 0,114 987 362 8
66	5,521 792 018	− 2,329 506 007	0,616 271 430 5	− 0,109 338 479 6
67	5,507 880 036	− 2,287 888 630	0,595 804 330 8	− 0,104 029 327 6
68	5,493 192 356	− 2,247 215 055	0,576 208 988 4	− 0,099 035 919 88
69	5,477 788 667	− 2,207 467 075	0,557 441 180 5	− 0,094 336 199 78
70	5,461 724 771	− 2,168 626 012	0,539 459 207 0	− 0,089 909 867 83
71	5,445 052 839	− 2,130 672 850	0,522 223 737 8	− 0,085 738 225 60
72	5,427 821 659	− 2,093 588 354	0,505 697 670 1	− 0,081 804 034 87
73	5,410 076 857	− 2,057 353 171	0,489 845 993 1	− 0,078 091 390 31
74	5,391 861 111	− 2,021 947 917	0,474 635 661 2	− 0,074 585 603 90
75	5,373 214 340	− 1,987 353 249	0,460 035 474 3	− 0,071 273 101 65
76	5,354 173 892	− 1,953 549 933	0,446 015 966 5	− 0,068 141 328 22
77	5,334 774 712	− 1,920 518 896	0,432 549 300 9	− 0,065 178 661 79
78	5,315 049 494	− 1,888 241 268	0,419 609 170 6	− 0,062 374 336 17
79	5,295 028 837	− 1,856 698 423	0,407 170 706 9	− 0,059 718 370 34
80	5,274 741 368	− 1,825 872 012	0,395 210 392 2	− 0,057 201 504 14
81	5,254 213 889	− 1,795 743 987	0,383 705 980 2	− 0,054 815 140 03
82	5,233 471 471	− 1,766 296 621	0,372 636 418 0	− 0,052 551 289 72
83	5,212 537 584	− 1,737 512 528	0,361 981 776 7	− 0,050 402 525 87
84	5,191 434 194	− 1,709 374 674	0,351 723 183 9	− 0,048 361 937 79
85	5,170 181 853	− 1,681 866 386	0,341 842 761 4	− 0,046 423 091 05
86	5,148 799 792	− 1,654 971 362	0,332 323 566 6	− 0,044 579 990 64
87	5,127 306 014	− 1,628 673 675	0,323 149 538 7	− 0,042 827 047 30
88	5,105 717 359	− 1,602 957 776	0,314 305 446 2	− 0,041 159 046 52
89	5,084 049 580	− 1,577 808 490	0,305 776 839 2	− 0,039 571 120 37
90	5,062 317 404	− 1,553 211 022	0,297 550 004 2	− 0,038 058 721 46
91	5,040 534 631	− 1,529 150 955	0,289 611 923 4	− 0,036 617 599 51
92	5,018 714 131	− 1,505 614 239	0,281 950 232 1	− 0,035 243 779 01
93	4,996 867 965	− 1,482 587 198	0,274 553 184 9	− 0,033 933 539 70
94	4,975 007 387	− 1,460 056 516	0,267 409 618 2	− 0,032 683 397 79
95	4,953 142 919	− 1,438 009 235	0,260 508 919 3	− 0,031 490 089 15
96	4,931 284 389	− 1,416 432 750	0,253 840 994 6	− 0,030 350 553 71
97	4,909 440 973	− 1,395 314 803	0,247 396 241 6	− 0,029 261 921 05
98	4,887 621 234	− 1,374 643 472	0,241 165 521 5	− 0,028 221 497 19
99	4,865 833 166	− 1,354 407 170	0,235 140 133 7	− 0,027 226 752 32
100	4,844 084 216	− 1,334 594 631	0,229 311 792 3	− 0,026 275 309 53

N	C_{75}	C_{76}	C_{77}
8	1,846 153 846	- 4.0	8,0
9	3.115 384 615	- 4.5	4,5
10	3,665 158 371	- 3,970 588 235	2,647 058 824
11	3,733 031 674	- 3,235 294 118	1,617 647 059
12	3,536 556 323	- 2,554 179 567	1,021 671 827
13	3,218 266 254	- 1,992 260 062	0,664 086 687 3
14	2,860 681 114	- 1.549 535 604	0,442 724 458 2
15	2,507 739 938	- 1.207 430 341	0.301 857 585 2
16	2,180 643 424	- 0,944 945 483 9	0,209 987 885 3
17	1,887 871 853	- 0,743 707 093 8	0.148 741 418 8
18	1,631 121 282	- 0,589 016 018 4	0,107 093 821 5
19	1,408 695 652	- 0,469 565 217 3	0,078 260 869 56
20	1.217 391 304	- 0.376 811 594 2	0,057 971 014 50
21	1,053 511 706	- 0,304 347 826 1	0,043 478 260 87
22	0,913 389 159 1	- 0,247 376 311 9	0,032 983 508 25
23	0,793 633 952 2	- 0.202 298 850 6	0,025 287 356 32
24	0,691 229 571 3	- 0.166 407 119 0	0,019 577 308 12
25	0,603 555 232 3	- 0,137 652 947 7	0,015 294 771 97
26	0,528 364 849 8	- 0,114 479 050 8	0,012 050 426 40
27	0,463 750 572 6	- 0,095 694 562 60	0,009 569 456 260
28	0,408 100 503 9	- 0.080 383 432 59	0,007 655 565 009
29	0.360 056 926 0	- 0.067 836 812 15	0,006 166 982 923
30	0.318 477 870 6	- 0,057 502 948 86	0,005 000 256 422
31	0,282 403 146 2	- 0,048 949 878 67	0,004 079 156 556
32	0,251 025 018 8	- 0,041 837 503 14	0,003 347 000 251
33	0,223 663 291 8	- 0,035 896 577 69	0,002 761 275 207
34	0,199 744 328 1	- 0,030 912 812 68	0,002 289 837 976
35	0,178 783 503 6	- 0,026 714 776 39	0,001 908 198 314
36	0,160 370 584 6	- 0,023 164 640 00	0,001 597 561 379
37	0,144 157 570 9	- 0,020 151 058 30	0,001 343 403 887
38	0,129 848 597 2	- 0,017 583 664 21	0,001 134 429 949
39	0,117 191 546 0	- 0,015 388 788 87	0,000 961 799 304 5
40	0,105 971 078 8	- 0.013 506 117 89	0,000 818 552 599 6
41	0,096 002 839 75	- 0,011 886 065 87	0,000 699 180 345 5
42	0,087 128 627 67	- 0,010 487 705 18	0,000 599 297 439 0
43	0,079 212 369 51	- 0,009 277 124 357	0,000 515 395 797 6
44	0,072 136 754 77	- 0,008 226 121 158	0,000 444 655 197 7
45	0,065 800 418 23	- 0,007 311 157 581	0,000 384 797 767 4
46	0,060 115 575 74	- 0,006 512 520 705	0,000 333 975 420 8
47	0,055 006 037 25	- 0,005 813 646 213	0,000 290 682 310 7
48	0,050 405 532 31	- 0,005 200 570 794	0,000 253 686 380 2
49	0,046 256 296 42	- 0,004 661 487 236	0,000 221 975 582 7
50	0.042 507 874 74	- 0,004 186 381 603	0,000 194 715 423 4

N	C_{75}	C_{76}	C_{77}
51	0,039 116 107 67	-0,003 766 736 295	0,000 171 215 286 1
52	0,036 042 268 70	-0,003 395 286 182	0,000 150 901 608 1
53	0,033 252 330 13	-0,003 065 817 671	0,000 133 296 420 5
54	0,030 716 336 31	-0,002 773 002 583	0,000 118 000 109 9
55	0,028 407 867 66	-0,002 512 260 405	0,000 104 677 516 9
56	0,026 303 581 16	-0,002 279 643 701	0,000 093 046 681 66
57	0,024 382 815 83	-0,002 071 742 521	0,000 082 869 700 85
58	0,022 627 253 09	-0,001 885 604 424	0,000 073 945 271 54
59	0,021 020 624 01	-0,001 718 667 372	0,000 066 102 591 23
60	0,019 548 456 31	-0,001 568 703 284	0,000 059 196 350 34
61	0,018 197 855 42	-0,001 433 770 427	0,000 053 102 608 42
62	0,016 957 314 55	-0,001 312 173 150	0,000 047 715 387 27
63	0,015 816 549 75	-0,001 202 427 759	0,000 042 943 848 54
64	0,014 766 356 31	-0,001 103 233 517	0,000 038 709 947 97
65	0,013 798 483 54	-0,001 013 447 944	0,000 034 946 480 82
66	0,012 905 525 46	-0,000 932 065 727 8	0,000 031 595 448 40
67	0,012 080 825 14	-0,000 858 200 693 1	0,000 028 606 689 77
68	0,011 318 390 84	-0,000 791 070 327 8	0,000 025 936 732 06
69	0,010 612 822 48	-0,000 729 982 498 3	0,000 023 547 822 53
70	0,009 959 246 898	-0,000 674 324 008 7	0,000 021 407 111 39
71	0,009 353 260 975	-0,000 623 550 731 7	0,000 019 485 960 36
72	0,008 790 881 359	-0,000 577 179 079 1	0,000 017 759 356 28
73	0,008 268 500 140	-0,000 534 778 516 0	0,000 016 205 412 61
74	0,007 782 845 624	-0,000 495 965 652 5	0,000 014 804 944 85
75	0,007 330 947 598	-0,000 460 397 675 3	0,000 013 541 108 10
76	0,006 910 106 524	-0,000 427 768 499 1	0,000 012 399 086 93
77	0,006 517 866 179	-0,000 397 804 039 1	0,000 011 365 829 69
78	0,006 151 989 321	-0,000 370 258 616 5	0,000 010 429 820 18
79	0,005 810 436 033	-0,000 344 911 728 0	0,000 009 580 881 333
80	0,005 491 344 397	-0,000 321 565 212 5	0,000 008 810 005 821
81	0,005 193 013 266	-0,000 300 040 766 5	0,000 008 109 209 905
82	0,004 913 886 831	-0,000 280 177 757 9	0,000 007 471 406 878
83	0,004 652 540 849	-0,000 261 831 303 2	0,000 006 890 297 453
84	0,004 407 670 279	-0,000 244 870 571 1	0,000 006 360 274 573
85	0,004 178 078 195	-0,000 229 177 284 9	0,000 005 876 340 639
86	0,003 962 665 835	-0,000 214 644 399 4	0,000 005 434 035 427
87	0,003 760 423 665	-0,000 201 174 928 6	0,000 005 029 373 215
88	0,003 570 423 313	-0,000 188 680 906 8	0,000 004 658 787 822
89	0,003 391 810 318	-0,000 177 082 466 4	0,000 004 319 084 546
90	0,003 223 797 583	-0,000 166 307 018 2	0,000 004 007 398 028
91	0,003 065 659 494	-0,000 156 288 523 2	0,000 003 721 155 314
92	0,002 916 726 539	-0,000 146 966 841 1	0,000 003 458 043 320
93	0,002 776 380 521	-0,000 138 287 152 4	0,000 003 215 980 288
94	0,002 644 050 158	-0,000 130 199 439 6	0,000 002 993 090 565
95	0,002 519 207 132	-0,000 122 658 025 1	0,000 002 787 682 389
96	0,002 401 362 491	-0,000 115 621 157 0	0,000 002 598 228 247
97	0,002 290 063 387	-0,000 109 050 637 5	0,000 002 423 347 499
98	0,002 184 890 105	-0,000 102 911 490 5	0,000 002 261 790 999
99	0,002 085 453 369	-0,000 097 171 662 36	0,000 002 112 427 443
100	0,001 991 391 880	-0,000 091 801 753 35	0,000 001 974 231 255

TABLE IV

The values of the binomial coefficients.

x	$\binom{x}{2}$	$\binom{x}{3}$	$\binom{x}{4}$	$\binom{x}{5}$	$\binom{x}{6}$	x
4	6	4	1			4
5	10	10	5	1		5
6	15	20	15	6	1	6
7	21	35	35	21	7	7
8	28	56	70	56	28	8
9	36	84	126	126	84	9
10	45	120	210	252	210	10
11	55	165	330	462	462	11
12	66	220	495	792	924	12
13	78	286	715	1 287	1 716	13
14	91	364	1 001	2 002	3 003	14
15	105	455	1 365	3 003	5 005	15
16	120	560	1 820	4 368	8 008	16
17	136	680	2 380	6 188	12 376	17
18	153	816	3 060	8 568	18 564	18
19	171	969	3 876	11 628	27 132	19
20	190	1 140	4 845	15 504	38 760	20
21	210	1 330	5 985	20 349	54 264	21
22	231	1 540	7 315	26 334	74 613	22
23	253	1 771	8 855	33 649	100 947	23
24	276	2 024	10 626	42 504	134 596	24
25	300	2 300	12 650	53 130	177 100	25
26	325	2 600	14 950	65 780	230 230	26
27	351	2 925	17 550	80 730	296 010	27
28	378	3 276	20 475	98 280	376 740	28
29	406	3 654	23 751	118 755	475 020	29
30	435	4 060	27 405	142 506	593 775	30
31	465	4 495	31 465	169 911	736 281	31
32	496	4 960	35 960	201 376	906 192	32
33	528	5 456	40 920	237 336	1 107 568	33
34	561	5 984	46 376	278 256	1 344 904	34
35	595	6 545	52 360	324 632	1 623 160	35
36	630	7 140	58 905	376 992	1 947 792	36
37	666	7 770	66 045	435 897	2 324 784	37
38	703	8 436	73 815	501 942	2 760 681	38
39	741	9 139	82 251	575 757	3 262 623	39
40	780	9 880	91 390	658 008	3 838 380	40
41	820	10 660	101 270	749 398	4 496 388	41
42	861	11 480	111 930	850 668	5 245 786	42
43	903	12 341	123 410	962 598	6 096 454	43
44	946	13 244	135 751	1 086 008	7 059 052	44
45	990	14 190	148 995	1 221 759	8 145 060	45
46	1 035	15 180	163 185	1 370 754	9 366 819	46
47	1 081	16 215	178 365	1 533 939	10 737 573	47
48	1 128	17 296	194 580	1 712 304	12 271 512	48
49	1 176	18 424	211 876	1 906 884	13 983 816	49
50	1 225	19 600	230 300	2 118 760	15 890 700	50
51	1 275	20 825	249 900	2 349 060	18 009 460	51
52	1 326	22 100	270 725	2 598 960	20 358 520	52
53	1 378	23 426	292 825	2 869 685	22 957 480	53
54	1 431	24 804	316 251	3 162 510	25 827 165	54
55	1 485	26 235	341 055	3 478 761	28 989 675	55

x	$\binom{x}{2}$	$\binom{x}{3}$	$\binom{x}{4}$	$\binom{x}{5}$	$\binom{x}{6}$	x
56	1 540	27 720	367 290	3 819 816	32 468 436	56
57	1 596	29 260	395 010	4 187 106	36 288 252	57
58	1 653	30 856	424 270	4 582 116	40 475 358	58
59	1 711	32 509	455 126	5 006 386	45 057 474	59
60	1 770	34 220	487 635	5 461 512	50 063 860	60
61	1 830	35 990	521 855	5 949 147	55 525 372	61
62	1 891	37 820	557 845	6 471 002	61 474 519	62
63	1 953	39 711	595 665	7 028 847	67 945 521	63
64	2 016	41 664	635 376	7 624 512	74 974 368	64
65	2 080	43 680	677 040	8 259 888	82 598 880	65
66	2 145	45 760	720 720	8 936 928	90 858 768	66
67	2 211	47 905	766 480	9 657 648	99 795 696	67
68	2 278	50 116	814 385	10 424 128	109 453 344	68
69	2 346	52 394	864 501	11 238 513	119 877 472	69
70	2 415	54 740	916 895	12 103 014	131 115 985	70
71	2 485	57 155	971 635	13 019 909	143 218 999	71
72	2 556	59 640	1 028 790	13 991 544	156 238 908	72
73	2 628	62 196	1 088 430	15 020 334	170 230 452	73
74	2 701	64 824	1 150 626	16 108 764	185 250 786	74
75	2 775	67 525	1 215 450	17 259 390	201 359 550	75
76	2 850	70 300	1 282 975	18 474 840	218 618 940	76
77	2 926	73 150	1 353 275	19 757 815	237 093 780	77
78	3 003	76 076	1 426 425	21 111 090	256 851 595	78
79	3 081	79 079	1 502 501	22 537 515	277 962 685	79
80	3 160	82 160	1 581 580	24 040 016	300 500 200	80
81	3 240	85 320	1 663 740	25 621 596	324 540 216	81
82	3 321	88 560	1 749 060	27 285 336	350 161 812	82
83	3 403	91 881	1 837 620	29 034 396	377 447 148	83
84	3 486	95 284	1 929 501	30 872 016	406 481 544	84
85	3 570	98 770	2 024 785	32 801 517	437 353 560	85
86	3 655	102 340	2 123 555	34 826 302	470 155 077	86
87	3 741	105 995	2 225 895	36 949 857	504 981 379	87
88	3 828	109 736	2 331 890	39 175 752	541 931 236	88
89	3 916	113 564	2 441 626	41 507 642	581 106 988	89
90	4 005	117 480	2 555 190	43 949 268	622 614 630	90
91	4 095	121 485	2 672 670	46 504 458	666 563 898	91
92	4 186	125 580	2 794 155	49 177 128	713 068 356	92
93	4 278	129 766	2 919 735	51 971 283	762 245 484	93
94	4 371	134 044	3 049 501	54 891 018	814 216 767	94
95	4 465	138 415	3 183 545	57 940 519	869 107 785	95
96	4 560	142 880	3 321 960	61 124 064	927 048 304	96
97	4 656	147 440	3 464 840	64 446 024	988 172 368	97
98	4 753	152 096	3 612 280	67 910 864	1 052 618 392	98
99	4 851	156 849	3 764 376	71 523 144	1 120 529 256	99
100	4 950	161 700	3 921 225	75 287 520	1 192 052 400	100
101	5 050	166 650	4 082 925	79 208 745	1 267 339 920	101
102	5 151	171 700	4 249 575	83 291 670	1 346 548 665	102
103	5 253	176 851	4 421 275	87 541 245	1 429 840 335	103
104	5 356	182 104	4 598 126	91 962 520	1 517 381 580	104
105	5 460	187 460	4 780 230	96 560 646	1 609 344 100	105
106	5 565	192 920	4 967 690	101 340 876	1 705 904 746	106
107	5 671	198 485	5 160 610	106 308 566	1 807 245 622	107
108	5 778	204 156	5 359 095	111 469 176	1 913 554 188	108
109	5 886	209 934	5 563 251	116 828 271	2 025 023 364	109
110	5 995	215 820	5 773 185	122 391 522	2 141 851 636	110

The values of the binomial coefficients.

x	$\binom{x}{7}$	$\binom{x}{8}$	$\binom{x}{9}$	$\binom{x}{10}$	x
4					4
5					5
6					6
7	1				7
8	8	1			8
9	36	9	1		9
10	120	45	10	1	10
11	330	165	55	11	11
12	792	495	220	66	12
13	1 716	1 287	715	286	13
14	3 432	3 003	2 002	1 001	14
15	6 435	6 435	5 005	3 003	15
16	11 440	12 870	11 440	8 008	16
17	19 448	24 310	24 310	19 448	17
18	31 824	43 758	48 620	43 758	18
19	50 388	75 582	92 378	92 378	19
20	77 520	125 970	167 960	184 756	20
21	116 280	203 490	293 930	352 716	21
22	170 544	319 770	497 420	646 646	22
23	245 157	490 314	817 190	1 144 066	23
24	346 104	735 471	1 307 504	1 961 256	24
25	480 700	1 081 575	2 042 975	3 268 760	25
26	657 800	1 562 275	3 124 550	5 311 735	26
27	888 030	2 220 075	4 686 825	8 436 285	27
28	1 184 040	3 108 105	6 906 900	13 123 110	28
29	1 560 780	4 292 145	10 015 005	20 030 010	29
30	2 035 800	5 852 925	14 307 150	30 045 015	30
31	2 629 575	7 888 725	20 160 075	44 352 165	31
32	3 365 856	10 518 300	28 048 800	64 512 240	32
33	4 272 048	13 884 156	38 567 100	92 561 040	33
34	5 379 616	18 156 204	52 451 256	131 128 140	34
35	6 724 520	23 535 820	70 607 460	183 579 396	35
36	8 347 680	30 260 340	94 143 280	254 186 856	36
37	10 295 472	38 608 020	124 403 620	348 330 136	37
38	12 620 256	48 903 492	163 011 640	472 733 756	38
39	15 380 937	61 523 748	211 915 132	635 745 396	39
40	18 643 560	76 904 685	273 438 880	847 660 528	40
41	22 481 940	95 548 245	350 343 565	1 121 099 408	41
42	26 978 328	118 030 185	445 891 810	1 471 442 973	42
43	32 224 114	145 008 513	563 921 995	1 917 334 783	43
44	38 320 568	177 232 627	708 930 508	2 481 256 778	44
45	45 379 620	215 553 195	886 163 135	3 190 187 286	45
46	53 524 680	260 932 815	1 101 716 330	4 076 350 421	46
47	62 891 499	314 457 495	1 362 649 145	5 178 066 751	47
48	73 629 072	377 348 994	1 677 106 640	6 540 715 896	48
49	85 900 584	450 978 066	2 054 455 634	8 217 822 536	49
50	99 884 400	536 878 650	2 505 433 700	10 272 278 170	50
51	115 775 100	636 763 050	3 042 312 350	12 777 711 870	51
52	133 784 560	752 538 150	3 679 075 400	15 820 024 220	52
53	154 143 080	886 322 710	4 431 613 550	19 499 099 620	53
54	177 100 560	1 040 465 790	5 317 936 260	23 930 713 170	54
55	202 927 725	1 217 566 350	6 358 402 050	29 248 649 430	55

x	$\binom{x}{y}$
56	231 917 [illegible]
57	264 [illegible]
58	300 [illegible]
59	341 [illegible]
60	386 [illegible]
61	434 [illegible]
62	487 [illegible]
63	[illegible]
64	[illegible]
65	[illegible]
66	[illegible]
67	[illegible]
68	[illegible]
69	1 07[illegible]
70	1 [illegible]
71	1 [illegible]
72	1 [illegible]
73	1 [illegible]
74	1 [illegible]
75	1 [illegible]
76	2 [illegible]
77	2 [illegible]
78	2 [illegible]
79	2 [illegible]
80	3 [illegible]
81	3 [illegible]
82	3 [illegible]
83	4 [illegible]
84	4 [illegible]
85	4 [illegible]
86	5 [illegible]
87	5 [illegible]
88	6 [illegible]
89	6 [illegible]
90	7 471 [illegible]
91	8 [illegible]
92	8 [illegible]
93	9 [illegible]
94	10 [illegible]
95	11 [illegible]
96	11 [illegible]
97	12 [illegible]
98	13 884 [illegible]
99	14 [illegible]
100	16 [illegible]
101	17 [illegible]
102	18 [illegible]
103	19 [illegible]
104	21 243 [illegible]
105	22 760 [illegible]
106	24 370 [illegible]
107	26 [illegible]
108	27 [illegible]
109	29 796 [illegible]
110	31 [illegible]

CONCERNING THE LIMITS OF A MEASURE OF SKEWNESS

By RAYMOND GARVER
University of California at Los Angeles

In a recent note in the Annals of Mathematical Statistics,* Hotelling and Solomons devised an ingenious method of showing that the measure of skewness s defined by the equation

$$S = \frac{mean - median}{standard\ deviation}$$

cannot be greater than unity in absolute value. I am venturing to offer another proof of the same fact, which seems to me to be of interest because it employs an important and well-known algebraic inequality.

With Hotelling and Solomons, I shall assume that we are concerned with n readings, or x's, with median zero and mean $\bar{x}$, where $\bar{x}$ of course is $\Sigma x/n$ We may show that the absolute value of s cannot be greater than one by showing that $1/s^2$ is not less than one. Making obvious substitutions, we must then show that

$$\frac{n\Sigma x^2}{(\Sigma x)^2} \geqq 2.$$

Now according to a known theorem if $a, b, \cdots, k$ are n positive numbers, and if m is a number not lying between zero and one, then

$$\frac{a^m + b^m + \cdots + k^m}{n} \geqq \left(\frac{a+b+\cdots+k}{n}\right)^m.$$

---

*Vol. 3, no. 2, May, 1932, 141-2.

While the proof of this theorem is given in Chrystal, we shall outline a (simplified) proof for the case $m=2$, to make this note self-contained. For any number r we obviously have $(r-1)^2 \geqq 0$. Now let r equal $na/_{a+b+\cdots+k}$, $nb/_{a+b+\cdots+k}$, $\cdots$, $nk/_{a+b+\cdots+k}$ in turn. The first of these gives

$$\frac{n^2a^2}{(a+b+\cdots+k)^2} - \frac{2na}{(a+b+\cdots+k)} + 1 \geqq 0,$$

while the others give similar inequalities. Summing these inequalities we have

$$\frac{n^2(a^2+b^2+\cdots+k^2)}{(a+b+\cdots k)^2} - 2n + n \geqq 0,$$

which is Chrystal's theorem,[*] for $m=2$. The proof shows that some of the numbers $a, b, \cdots, k$ can be zero; in fact, some can be negative, provided $a+b+\cdots+k$ is not zero.

Now, suppose we have an odd number of readings, say $n=2s+1$. Since the median reading is zero, there are s non-negative readings, which we shall now call y's, and s non-positive readings, which we shall call z's. We have at once, by the above,

$$\frac{s\Sigma y^2}{(\Sigma y)^2} \geqq 1,$$

$$\frac{s\Sigma z^2}{(\Sigma z)^2} \geqq 1,$$

It follows immediately that

$$s\left(\frac{\Sigma y^2+\Sigma z^2}{(\Sigma y)^2+(\Sigma z)^2}\right) \geqq 1$$

---

[*] Chrystal, Algebra, Part II, 2nd ed., 1922, p. 49.

and, since $n = 2s+1$ that

$$n\left(\frac{\Sigma y^2+\Sigma z^2}{(\Sigma y)^2+(\Sigma z)^2}\right) > 2.$$

Finally,

$$n\frac{\Sigma x^2}{(\Sigma x)^2} = n\frac{\Sigma y^2+\Sigma z^2}{(\Sigma y+\Sigma z)^2} = n\frac{\Sigma y^2+\Sigma z^2}{(\Sigma y)^2+(\Sigma z)^2+2(\Sigma y)(\Sigma z)} > 2,$$

since $2(\Sigma y)(\Sigma z)$ is certainly not positive.

This proof is valid unless all the y's are zero or all the z's are zero. Suppose the latter is the case. Then

$$\frac{n\Sigma x^2}{(\Sigma x)^2} = \frac{n\Sigma y^2}{(\Sigma y)^2} > 2\left(\frac{s\Sigma y^2}{(\Sigma y)^2}\right) > 2.$$

If all the readings are zero our definition of s does not give a definite value.

If n is even, not odd, the proof may be modified by properly defining the median. In this case we can show again that

$$\frac{n\Sigma x^2}{(\Sigma x)^2} \geqq 2,$$

but the possibility of the equality cannot be ruled out.

Raymond Dawes

EDITORIAL.

Trapezoidal Rule for Computing Seasonal Indices.

The following method for computing seasonals is suggested by the Detroit Edison article on "*A Mathematical Theory of Seasonals*" that appeared in Vol. I, No. 1 of the *Annals*.

We shall likewise define "the seasonal index for any month as the ratio of the total of the variates for the month in question to the total that would have been experienced if neither accidental nor seasonal influences were present". that is, the seasonal index for the *i-th* month is

$$(1) \qquad s_i = \frac{\Sigma_0 y_i}{\Sigma \psi_i} \, .$$

The numerator presents no difficulties: the obstacle is met in determining the denominator, since $\psi(x)$ is the unknown function that is the consequence of only trend and cycle influences. According to accepted concepts the trend may be represented by some smooth analytic function, the cycle is a smooth though not a mathematically periodic function—but the seasonal and residual influences may inject all sorts of disturbances into a time series. We shall make but two further assumptions,—

(a) The smooth function $y=\psi(x)$, representing the combined effect of trend and cycle, may be approximated by the upper sides of a series of trapezoids as in figure (1). The area of each trapezoid is to equal the area under the function $\psi(x)$ limited by the common ordinates.

(b) Neither seasonal nor accidental influences affect annual totals. Thus we might assume that the seasonal activity in the production of coal does not affect the total coal mined within the year, but merely concentrates production within certain months and compensates this with a corresponding under-production in others. Although accidental disturbances within the year would merely attribute production to one month rather than the next, we must admit that if one of these months is the last of one year, and the other the first of the following year, such an accidental fluctuation will affect the annual totals. Usually, however, such perturbations represent but a small percent of the monthly production, and a negligible part of the annual total.

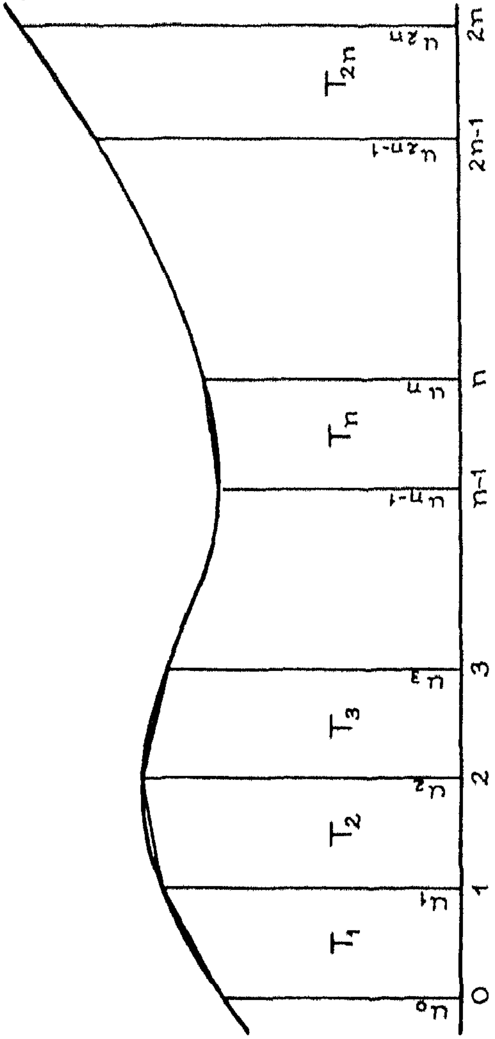

Fig. 1

Let us assume that we are dealing with $2n$ consecutive years: then by the above assumptions the area of the n-th trapezoid equals the total production for the n-th year, T_n .

Designating the ordinates corresponding to the n-th trapezoid by u_{n-1} and u_n , it follows since we are dealing with equal unit bases,

$$T_n = \frac{1}{2}(u_{n-1} + u_n), \qquad \text{that is}$$

$$u_n = 2T_n - u_{n-1}, \qquad \text{so that}$$

$$(2)\begin{cases} u_1 = 2T_1 - u_0 \\ u_2 = 2T_2 - u_1 = 2T_2 - 2T_1 + u_0 \\ u_3 = 2T_3 - u_2 = 2T_3 - 2T_2 + 2T_1 - u_0 \\ u_4 = 2T_4 - u_3 = 2T_4 - 2T_3 + 2T_2 - 2T_1 + u_0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ u_{2n-1} = 2T_{2n-1} - u_{2n-2} = 2T_{2n-1} - 2T_{2n-2} + \cdots + 2T_1 - u_0 \\ u_{2n} = 2T_{2n} - u_{2n-1} = 2T_{2n} - 2T_{2n-1} + \cdots + 2T_2 - 2T_1 + u_0 \end{cases}$$

By elementary geometry, the area corresponding to the twelve months of the n-th year are

$$\begin{aligned} M_1 &= \frac{1}{288}(23u_{n-1} + u_n) \\ M_2 &= \frac{1}{288}(21u_{n-1} + 3u_n) \\ M_3 &= \frac{1}{288}(19u_{n-1} + 5u_n) \\ &\cdots\cdots\cdots \\ M_{11} &= \frac{1}{288}(3u_{n-1} + 21u_n) \\ M_{12} &= \frac{1}{288}(u_{n-1} + 23u_n) \end{aligned}$$

According to definition, ΣV_i equals the sum of the values of M_i, one for each of the $2n$ years Thus.

$$
\begin{aligned}
\Sigma\psi_1 = \frac{1}{288}(&23u_0 + u_1 \\
&+ 23u_1 + u_2 \\
&+ 23u_2 + u_3 \\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
&+ 23u_{2n-2} + u_{2n-1} \\
&+ 23u_{2n-1} + u_{2n}),
\end{aligned}
$$

that is

$$
\Sigma\psi_1 = \frac{1}{288}\ 24(u_0 + u_1 + \cdots + u_{2n-1}) + (u_{2n} - u_0) \tag{3}
$$

and in precisely the same manner

$$
\Sigma\psi_2 = \frac{1}{288}\ 24(u_0 + u_1 + \cdots + u_{2n-1}) + 3(u_{2n} - u_0)
$$

$$
\Sigma\psi_3 = \frac{1}{288}\ 24(u_0 + u_1 + \cdots + u_{2n-1}) + 5(u_{2n} - u_0)
$$

But by (2)

$$
\begin{aligned}
u_0 + u_1 + u_2 + \cdots + u_{2n-1} &= 2T_1 + 2T_3 + 2T_5 + \cdots + 2T_{2n-3} + 2T_{2n-1} \\
&= 2\cdot O,
\end{aligned}
$$

where O designates the sum of the totals for the odd years.

Again,

$$
\begin{aligned}
u_{2n} - u_0 &= 2T_{2n} - 2T_{2n-1} + 2T_{2n-2} - \cdots + 2T_2 - 2T_1 \\
&= 2\cdot E - 2\cdot O,
\end{aligned}
$$

E representing the corresponding sum for the even years.

We have finally that

$$(4)\quad \begin{cases} \Sigma\psi_1 = \frac{1}{144}(23\cdot O + E) \\ \delta = \frac{1}{72}(E - O). \end{cases}$$

where δ represents the common difference $\Sigma\psi_{i+1} - \Sigma\psi_i$, which from equation (3) is seen to be

$$= \frac{1}{288}(2u_{2n} - 2u_0) = \frac{1}{72}(E - O).$$

It should be observed that we have not imposed the condition that the long time trend is a straight line—no matter what the law of growth may be, the assumption that the trend-cycle function, $\psi(x)$, may be approximated within each year by a secant line is alone responsible for the equal differences δ.

As an illustration let us compute the seasonals for the theoretical series presented in the Detroit Edison article, and reproduced below.

TABLE 1

THEORETICAL SERIES.

	1904	1905	1906	1907	1908	1909
Jan.	906	1662	1908	2030	1242	1714
Feb.	814	1582	1860	1855	1052	1831
Mar.	1138	1913	2052	2077	1283	1831
Apr.	1215	1976	2027	2088	1210	2077
May	1343	1892	2122	2043	1203	2143
June	1236	1700	1672	2093	1166	2058
July	1254	2092	2041	2060	1240	2320
Aug.	1702	1757	1846	2163	1334	2413
Sept.	1457	1906	2102	2262	1279	2502
Oct.	1564	1899	2304	1946	1364	2643
Nov.	1596	1611	2303	1475	1341	2378
Dec.	1836	2163	2170	1153	1564	2595
Total	16061	22153	24407	23245	15278	26505

TABLE 1 *Continued*

	1910	1911	1912	1913	1914	1915
Jan.	2392	1933	2052	2554	2041	1687
Feb.	2514	1746	2061	2330	1780	1532
Mar.	2417	1895	2267	2554	2194	1906
Apr.	2830	1865	2490	2800	2037	1796
May	2702	2167	2419	2845	2156	2539
June	2475	1732	2571	2696	1730	2674
July	2211	1742	2657	2314	1577	2566
Aug.	2249	2011	2469	2525	1649	2661
Sept.	2108	1976	2591	2377	1652	2952
Oct.	2203	2144	2516	2850	1753	3342
Nov.	1875	2168	2389	2494	1585	3093
Dec.	1899	2092	2435	2150	1779	3182
Total	27875	23471	28917	30489	21933	29930

Here

$$E = 155\ 793$$
$$O = 134\ 471$$
$$\Sigma \Psi_1 = 22\ 559\ 90$$
$$\delta = 296.14$$

Column 1 of table 2 is obtained by adding the items of table 1 horizontally. Column 2 is found by repeated adding the common difference, $\delta = 296.14$ to the value 22 559.90.

TABLE 2
SEASONALS BY TRAPEZOIDAL RULE.

Month	$\Sigma_0 y_i$	$\Sigma \phi_i$	S
Jan.	22 121	22 560	.981
Feb.	20 957	22 856	.917
Mar.	23 527	23 152	1.016
Apr.	24 411	23 448	1.041
May	25 574	23 744	1.077
June	23 803	24 041	.990
July	24 074	24 337	1.015
Aug.	24 779	24 633	1.006
Sept.	25 164	24 929	1.009
Oct.	26 528	25 225	1.052
Nov.	24 308	25 521	.952
Dec.	25 018	25 817	.969
Total	290 264	290 263	12.025

The following table presents the mean and standard errors for the results obtained by Link-relative method, the Interpolation method suggested in the Detroit Edison article, and the Trapezoidal method. Obviously, the last requires by far the least time in application.

TABLE 3

Method	$M.D$	σ
Link Relative	0277	.0338
Interpolation	.0269	.0337
Trapezoidal	.0227	.0255

For weekly indices the formulae by the trapezoidal rule are

$$\Sigma \psi_1 = \frac{1}{52^2}(103\ O + E)$$

$$\delta = \frac{1}{2\ 52^2}(E - O)$$

If one has data for $2n+1$ years, he may either disregard the most distant year and then compute seasonals for the final $2n$ years, or he may combine the results for the first $2n$ years (neglecting the last year) and the results for the last $2n$ years (neglecting the first year). This yields an almost equally simple formula.

Applying the Trapezoidal rule to successive overlapping periods will reveal the presence of a shifting seasonal. The change in the seasonal for Automobile production, caused by the advent of good roads and closed models, affords a good example.